a CAS or computer grapher to perform the following steps.

a) Plot the function $y = f(x)$ together with the function $y = f(ax)$ for $a = 2, 3,$ and 10 over the specified interval. Describe what happens to the graph as a increases through positive values.

b) Plot the function $y = f(x)$ and $y = f(ax)$ for the negative values $a = -2, -3$. What happens to the graph in this situation?

c) Plot the function $y = f(x)$ and $y = f(ax)$ for the fractional values $a = 1/2, 1/3, 1/4$. Describe what happens to the graph when $|a| < 1$.

83. $f(x) = \dfrac{5x}{x^2+4}, \quad [-10, 10]$

84. $f(x) = \dfrac{2x(x-1)}{x^2+1}, \quad [-3, 2]$

85. $f(x) = \dfrac{x+1}{2x^2+1}, \quad [-2, 2]$

86. $f(x) = \dfrac{x^4 - 4x^3 + 10}{x^2+4}, \quad [-1, 4]$

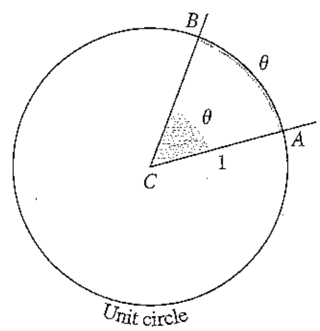
5 Trigonometric Functions

This section reviews radian measure, trigonometric functions, periodicity, and basic trigonometric identities.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations (Section 2.4).

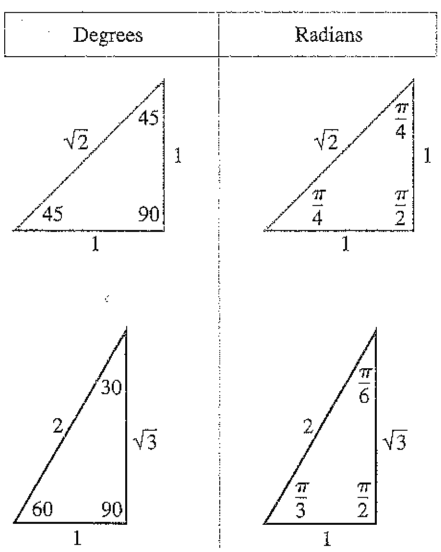
Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle ACB is the length of the arc AB.

The **radian measure** θ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by the following equation.

$$\pi \text{ radians} = 180°$$

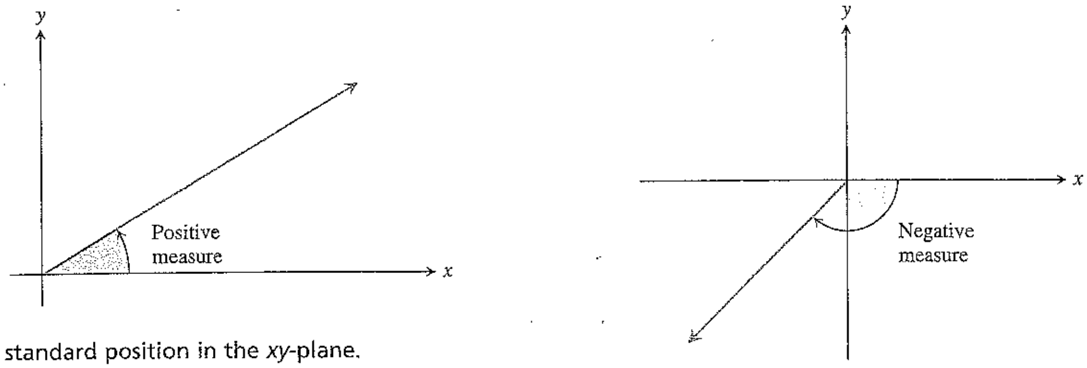


48 The angles of two common triangles, in degrees and radians.

EXAMPLE 1 *Conversions (Fig. 48)*

Convert $45°$ to radians: $\quad 45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\quad \dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30°$



49 Angles in standard position in the xy-plane.

Conversion formulas

1 degree $= \frac{\pi}{180}$ (≈ 0.02) radians

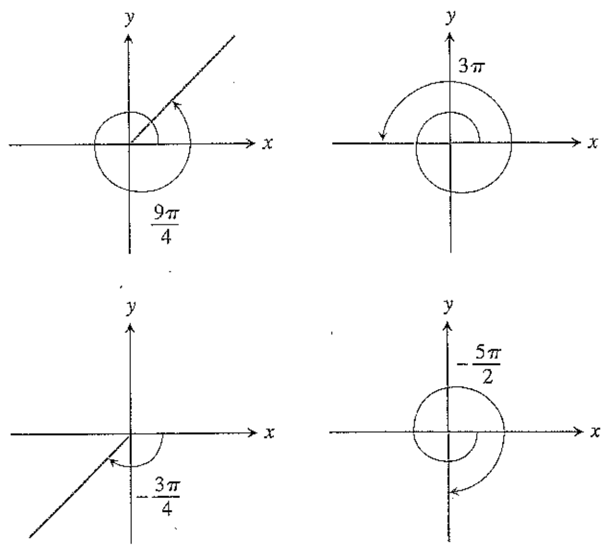
Degrees to radians: multiply by $\frac{\pi}{180}$

1 radian $= \frac{180}{\pi}$ (≈ 57) degrees

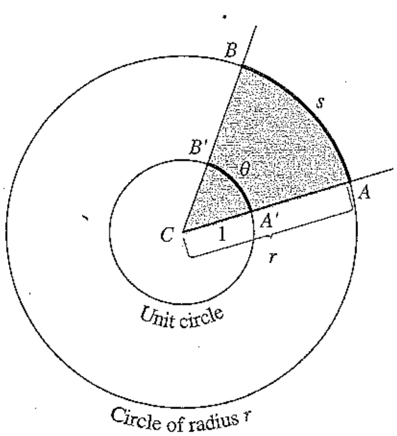
Radians to degrees: multiply by $\frac{180}{\pi}$

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig. 49). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

When angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Fig. 50).



50 Nonzero radian measures can be positive or negative.



51 The radian measure of angle ACB is the length θ of arc $A'B'$ on the unit circle centered at C. The value of θ can be found from any other circle as s/r.

There is a useful relationship between the length s of an arc AB on a circle of radius r and the radian measure θ of the angle the arc subtends at the circle's center C (Fig. 51). If we draw a unit circle with the same center C, the arc $A'B'$ cut by the angle will have length θ, by the definition of radian measure. From the similarity of the circular sectors ACB and $A'CB'$, we then have $s/r = \theta/1$.

Radian Measure and Arc Length

$$\frac{s}{r} = \theta, \qquad \text{or} \qquad s = r\theta$$

Notice that these equalities hold precisely because we are measuring the angle in radians.

Angle Convention: Use Radians

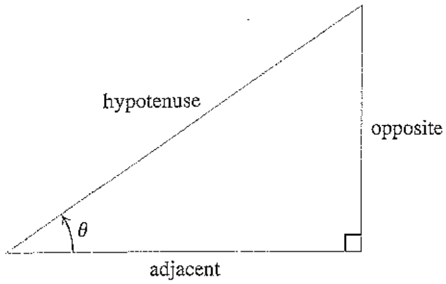
From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

EXAMPLE 2 Consider a circle of radius 8. (a) Find the central angle subtended by an arc of length 2π on the circle. (b) Find the length of an arc subtending a central angle of $3\pi/4$.

Solution

a) $\theta = \dfrac{s}{r} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$ **b)** $s = r\theta = 8\left(\dfrac{3\pi}{4}\right) = 6\pi$

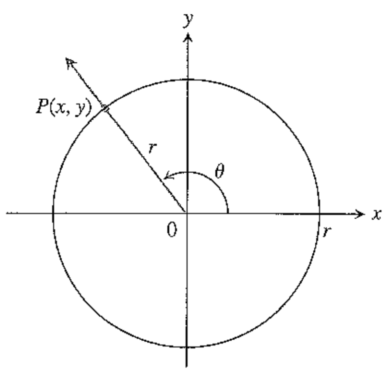
The Six Basic Trigonometric Functions



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

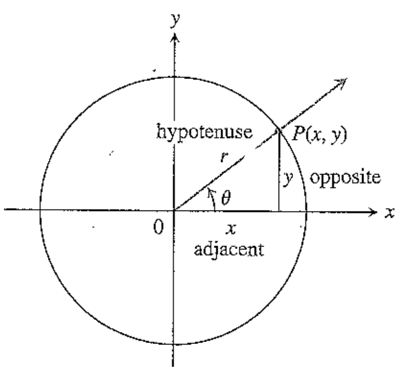
52 Trigonometric ratios of an acute angle.

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Fig. 52). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Fig. 53).



53 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

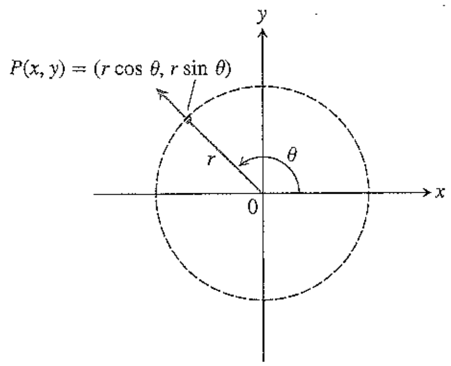
Sine:	$\sin\theta = \dfrac{y}{r}$	Cosecant:	$\csc\theta = \dfrac{r}{y}$
Cosine:	$\cos\theta = \dfrac{x}{r}$	Secant:	$\sec\theta = \dfrac{r}{x}$
Tangent:	$\tan\theta = \dfrac{y}{x}$	Cotangent:	$\cot\theta = \dfrac{x}{y}$



54 The new and old definitions agree for acute angles.

These extended definitions agree with the right-triangle definitions when the angle is acute (Fig. 54).

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means they are



55 The Cartesian coordinates of a point in the plane expressed in terms of r and θ.

not defined if θ is $\pm\pi/2,\ \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0,\ \pm\pi,\ \pm 2\pi, \ldots$.

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

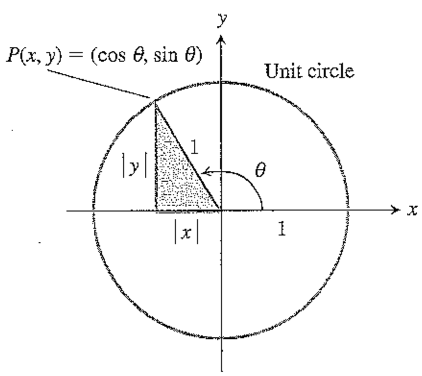
The coordinates of any point $P(x, y)$ in the plane can now be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Fig. 55). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

Values of Trigonometric Functions

If the circle in Fig. 53 has radius $r = 1$, the equations defining $\sin\theta$ and $\cos\theta$ become

$$\cos\theta = x, \qquad \sin\theta = y.$$



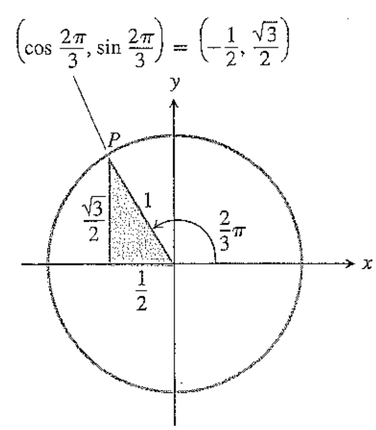
56 The acute reference triangle for an angle θ.

We can then calculate the values of the cosine and sine directly from the coordinates of P, if we happen to know them, or indirectly from the acute reference triangle made by dropping a perpendicular from P to the x-axis (Fig. 56). We read the magnitudes of x and y from the triangle's sides. The signs of x and y are determined by the quadrant in which the triangle lies.

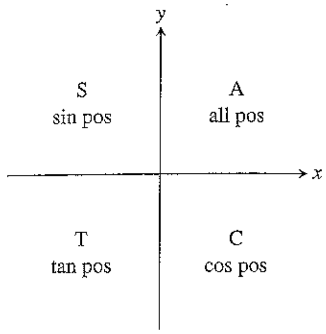
EXAMPLE 3 Find the sine and cosine of $2\pi/3$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 57).



57 The triangle for calculating the sine and cosine of $2\pi/3$ radians (Example 3).



58 The CAST rule.

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

$$\cos\frac{2\pi}{3} = x\text{-coordinate of } P = -\frac{1}{2}$$

$$\sin\frac{2\pi}{3} = y\text{-coordinate of } P = \frac{\sqrt{3}}{2}.$$

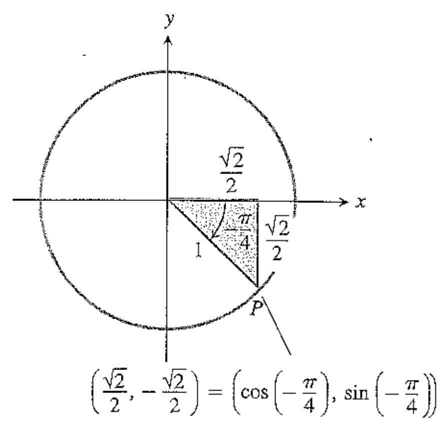
❑

A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule (Fig. 58).

EXAMPLE 4 Find the sine and cosine of $-\pi/4$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 59).



59 The triangle for calculating the sine and cosine of $-\pi/4$ radians (Example 4).

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

$$\cos\left(-\frac{\pi}{4}\right) = x\text{-coordinate of } P = \frac{\sqrt{2}}{2},$$

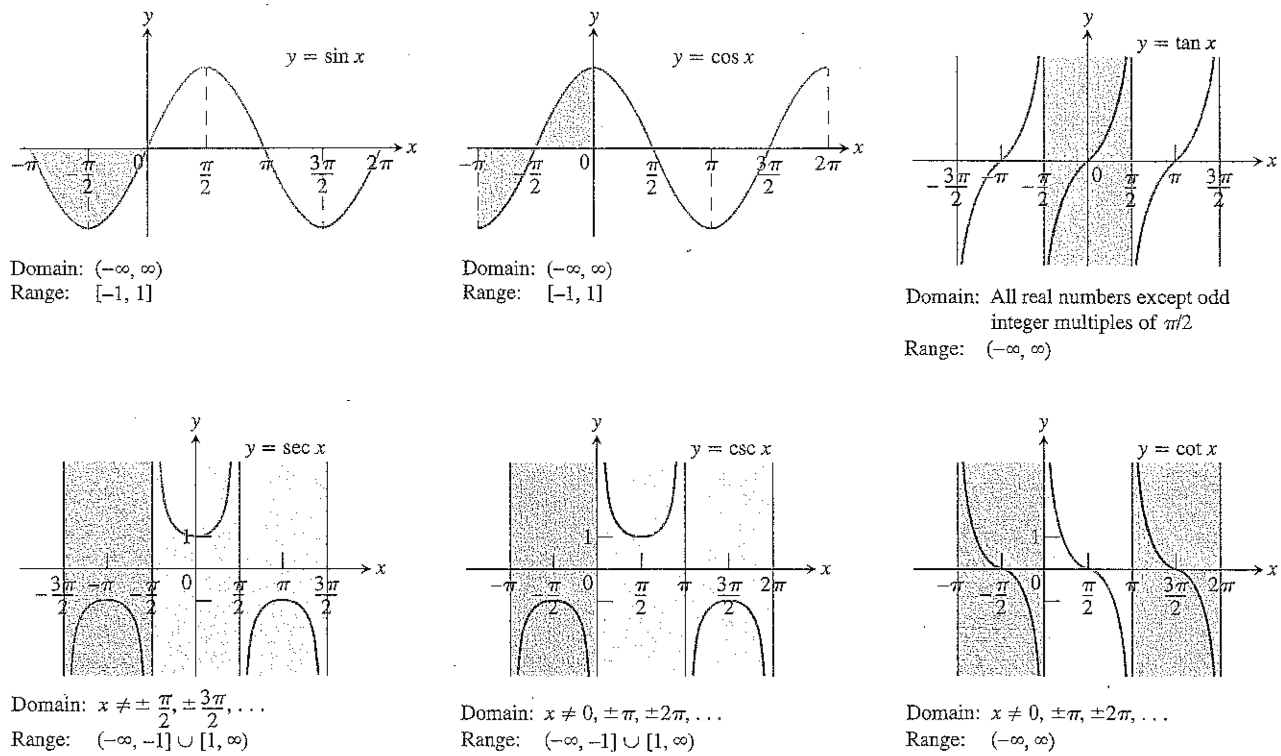
$$\sin\left(-\frac{\pi}{4}\right) = y\text{-coordinate of} P = -\frac{\sqrt{2}}{2}.$$

❑

Calculations similar to those in Examples 3 and 4 allow us to fill in Table 2.

Table 2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	135	180
θ (radians)	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π
$\sin\theta$	0	$-\sqrt{2}/2$	−1	$-\sqrt{2}/2$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{2}/2$	0
$\cos\theta$	−1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$-\sqrt{2}/2$	−1
$\tan\theta$	0	1		−1	0	$\sqrt{3}/3$	1	$\sqrt{3}$		−1	0



60 The graphs of the six basic trigonometric functions as functions of radian measure. Each function's periodicity shows clearly in its graph.

Graphs

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Fig. 60.

Periodicity

When an angle of measure x and an angle of measure $x + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric values. For example, $\cos(x + 2\pi) = \cos x$. Functions like the trigonometric functions whose values repeat at regular intervals are called periodic.

Definition

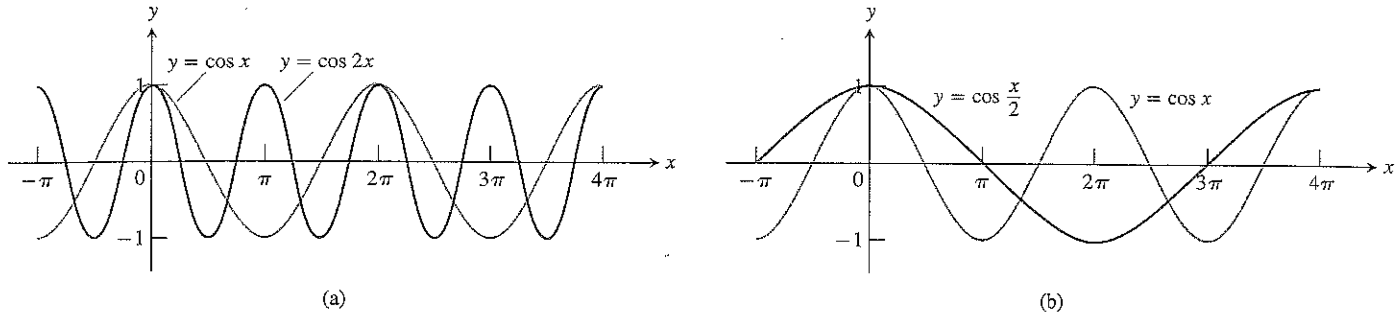
A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for all x. The smallest such value of p is the **period** of f.

Periods of trigonometric functions

Period π:	$\tan(x + \pi) = \tan x$
	$\cot(x + \pi) = \cot x$
Period 2π:	$\sin(x + 2\pi) = \sin x$
	$\cos(x + 2\pi) = \cos x$
	$\sec(x + 2\pi) = \sec x$
	$\csc(x + 2\pi) = \csc x$

As we can see in Fig. 60, the tangent and cotangent functions have period $p = \pi$. The other four functions have period 2π.

Figure 61 shows graphs of $y = \cos 2x$ and $y = \cos(x/2)$ plotted against the graph of $y = \cos x$. Multiplying x by a number greater than 1 speeds up a trigonometric function (increases the frequency) and shortens its period. Multiplying x by a positive number less than 1 slows a trigonometric function down and lengthens its period.



61 (a) Shorter period: cos 2*x*. (b) Longer period: cos (*x*/2)

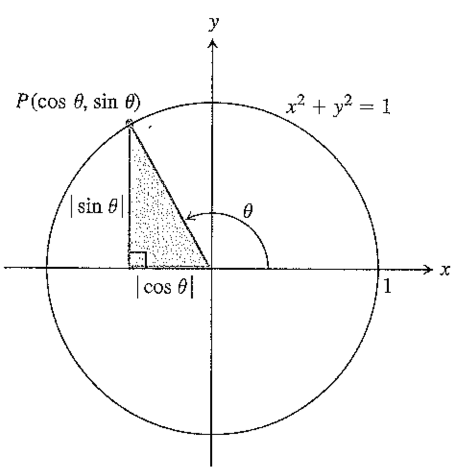
The importance of periodic functions stems from the fact that much of the behavior we study in science is periodic. Brain waves and heartbeats are periodic, as are household voltage and electric current. The electromagnetic field that heats food in a microwave oven is periodic, as are cash flows in seasonal businesses and the behavior of rotational machinery. The seasons are periodic—so is the weather. The phases of the moon are periodic, as are the motions of the planets. There is strong evidence that the ice ages are periodic, with a period of 90,000–100,000 years.

If so many things are periodic, why limit our discussion to trigonometric functions? The answer lies in a surprising and beautiful theorem from advanced calculus that says that every periodic function we want to use in mathematical modeling can be written as an algebraic combination of sines and cosines. Thus, once we learn the calculus of sines and cosines, we will know everything we need to know to model the mathematical behavior of periodic phenomena.

Even vs. Odd

The symmetries in the graphs in Fig. 60 reveal that the cosine and secant functions are even and the other four functions are odd:

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$



62 The reference triangle for a general angle θ.

Identities

Applying the Pythagorean theorem to the reference right triangle we obtain by dropping a perpendicular from the point $P(\cos\theta, \sin\theta)$ on the unit circle to the x-axis (Fig. 62) gives

$$\boxed{\cos^2\theta + \sin^2\theta = 1. \qquad (2)}$$

This equation, true for all values of θ, is probably the most frequently used identity in trigonometry.

Dividing Eq. (2) in turn by $\cos^2\theta$ and $\sin^2\theta$ gives the identities

$$1+\tan^2\theta = \sec^2\theta,$$
$$1+\cot^2\theta = \csc^2\theta.$$

You may recall the following identities from an earlier course.

All the trigonometric identities you will need in this book derive from Eqs. (2) and (3).

Angle Sum Formulas

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \sin(A+B) &= \sin A\cos B + \cos A\sin B\end{aligned} \tag{3}$$

These formulas hold for all angles A and B. There are similar formulas for $\cos(A-B)$ and $\sin(A-B)$ (Exercises 35 and 36).

Substituting θ for both A and B in the angle sum formulas gives two more useful identities:

Instead of memorizing Eqs. (3) you might find it helpful to remember Eqs. (4), and then recall where they came from.

Double-angle Formulas

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned} \tag{4}$$

Additional formulas come from combining the equations

$$\cos^2\theta+\sin^2\theta = 1, \qquad \cos^2\theta-\sin^2\theta=\cos 2\theta.$$

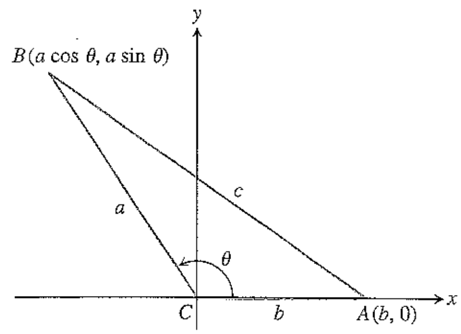
We add the two equations to get $2\cos^2\theta = 1+\cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1-\cos 2\theta$.

Additional Double-angle Formulas

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{5}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{6}$$

When θ is replaced by $\theta/2$ in Eqs. (5) and (6), the resulting formulas are called **half-angle** formulas. Some books refer to Eqs. (5) and (6) by this name as well.



63 The square of the distance between A and B gives the law of cosines.

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{7}$$

This equation is called the **law of cosines.**

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Fig. 63. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta,\ a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

Combining these equalities gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Exercises 5

Radians, Degrees, and Circular Arcs

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) 110°?

2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.

3. CALCULATOR You want to make an 80° angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?

4. CALCULATOR If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the table of function values shown on the following page. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \text{ in } \left[\frac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \text{ in } \left[0, \frac{\pi}{2}\right]$

9. $\cos x = \frac{1}{3}, \quad x \text{ in } \left[-\frac{\pi}{2}, 0\right]$

10. $\cos x = -\frac{5}{13}, \quad x \text{ in } \left[\frac{\pi}{2}, \pi\right]$

11. $\tan x = \frac{1}{2}, \quad x \text{ in } \left[\pi, \frac{3\pi}{2}\right]$

12. $\sin x = -\frac{1}{2}, \quad x \text{ in } \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$

14. $\sin(x/2)$

15. $\cos \pi x$

16. $\cos \frac{\pi x}{2}$

17. $-\sin \frac{\pi x}{3}$

18. $-\cos 2\pi x$

19. $\cos\left(x - \frac{\pi}{2}\right)$

20. $\sin\left(x + \frac{\pi}{2}\right)$

21. $\sin\left(x - \frac{\pi}{4}\right) + 1$

22. $\cos\left(x + \frac{\pi}{4}\right) - 1$

Graph the functions in Exercises 23–26 in the ts-plane (t-axis horizontal, s-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$

24. $s = -\tan \pi t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$

26. $s = \csc\left(\frac{t}{2}\right)$

27. GRAPHER
    a) Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.
    b) Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

28. GRAPHER Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Additional Trigonometric Identities

Use the angle sum formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the angle sum formulas? Do the results agree with something you already know?

Using the Angle Sum Formulas

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$

42. $\cos\left(\frac{3\pi}{2} + x\right)$

43. Evaluate $\sin \frac{7\pi}{12}$ as $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

44. Evaluate $\cos \dfrac{11\pi}{12}$ as $\cos\left(\dfrac{\pi}{4} + \dfrac{2\pi}{3}\right)$.

45. Evaluate $\cos \dfrac{\pi}{12}$.

46. Evaluate $\sin \dfrac{5\pi}{12}$.

Using the Double-angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \dfrac{\pi}{8}$

48. $\cos^2 \dfrac{\pi}{12}$

49. $\sin^2 \dfrac{\pi}{12}$

50. $\sin^2 \dfrac{\pi}{8}$

Theory and Examples

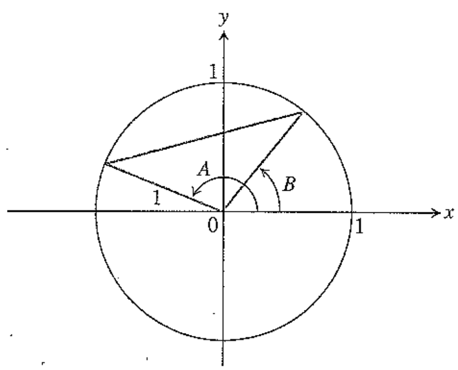
51. *The tangent sum formula.* The standard formula for the tangent of the sum of two angles is

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

52. *(Continuation of Exercise 51.)* Derive a formula for $\tan(A - B)$.

53. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



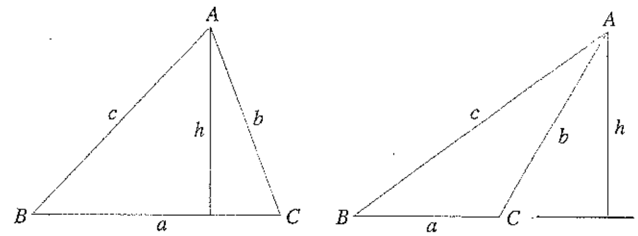
54. When applied to a figure similar to the one in Exercise 53, the law of cosines leads directly to the formula for $\cos(A + B)$. What is that figure and how does the derivation go?

55. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

56. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

57. *The law of sines.* The **law of sines** says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.

58. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 55). Find the sine of angle B using the law of sines.

59. CALCULATOR A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

60. *The approximation $\sin x \approx x$.* It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.7, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a) With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b) With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

c) *A quick radian mode check.* Is your calculator in radian mode? Evaluate $\sin x$ at a value of x near the origin, say $x = 0.1$. If $\sin x \approx x$, the calculator is in radian mode; if not, it isn't. Try it.

General Sine Curves

Figure 64 on the following page shows the graph of a **general sine function** of the form

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

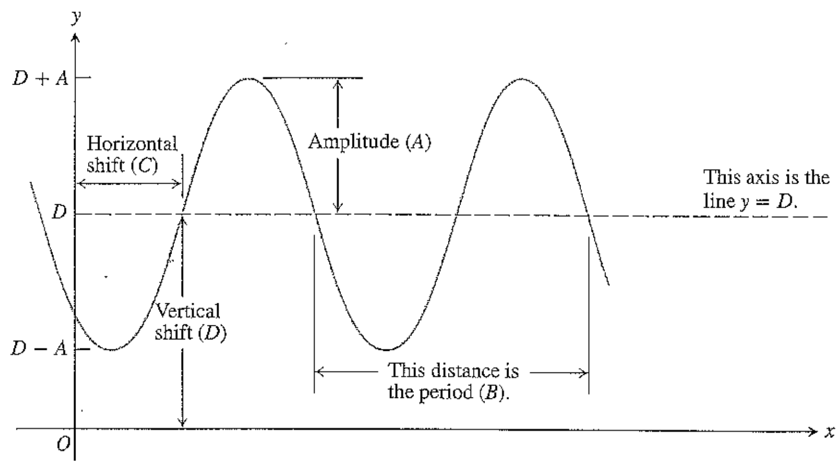
where $|A|$ is the *amplitude,* $|B|$ is the *period,* C is the *horizontal shift,* and D is the *vertical shift.* Identify A, B, C, and D for the sine functions in Exercises 61–64 and sketch their graphs.

61. $y = 2\sin(x + \pi) - 1$

62. $y = \dfrac{1}{2}\sin(\pi x - \pi) + \dfrac{1}{2}$

63. $y = -\dfrac{2}{\pi}\sin\left(\dfrac{\pi}{2}t\right) + \dfrac{1}{\pi}$

64. $y = \dfrac{L}{2\pi}\sin\dfrac{2\pi t}{L}, \quad L > 0$



64 The general sine curve

$$y = A\sin[(2\pi/B)(x - C)] + D,$$

shown for *A*, *B*, *C*, and *D* positive.

The Trans-Alaska Pipeline

The builders of the Trans-Alaska Pipeline used insulated pads to keep the heat from the hot oil in the pipeline from melting the permanently frozen soil beneath. To design the pads, it was necessary to take into account the variation in air temperature throughout the year. Figure 65 shows how we can use a general sine function, defined in the introduction to Exercises 61–64, to represent temperature data. The data points in the figure are plots of the mean air temperature for Fairbanks, Alaska, based on records of the National Weather Service from 1941 to 1970. The sine function used to fit the data is

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25,$$

where f is temperature in degrees Fahrenheit and x is the number of the day counting from the beginning of the year. The fit is remarkably good.

65. *Temperature in Fairbanks, Alaska.* Find the (a) amplitude, (b) period, (c) horizontal shift, and (d) vertical shift of the general sine function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

66. *Temperature in Fairbanks, Alaska.* Use the equation in Exercise 65 to approximate the answers to the following questions about the temperature in Fairbanks, Alaska, shown in Fig. 65. Assume that the year has 365 days.

a) What are the highest and lowest mean daily temperatures shown?

b) What is the average of the highest and lowest mean daily temperatures shown? Why is this average the vertical shift of the function?

CAS Explorations and Projects

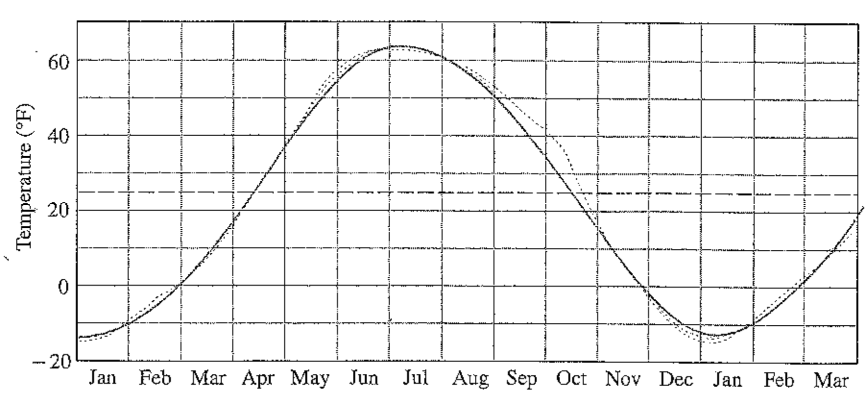
In Exercises 67–70, you will explore graphically the general sine function

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

67. *The period B.* Set the constants $A = 3$, $C = D = 0$.

a) Plot $f(x)$ for the values $B = 1,\ 3,\ 2\pi,\ 5\pi$ over the interval



65 Normal mean air temperature at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

(Source: "Is the Curve of Temperature Variation a Sine Curve?" by B. M. Lando and C. A. Lando, *The Mathematics Teacher,* 7:6, Fig. 2, p. 535 [September 1977].)

$-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b) What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

68. *The horizontal shift C.* Set the constants $A = 3$, $B = 6$, $D = 0$.

a) Plot $f(x)$ for the values $C = 0$, 1, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b) What happens to the graph for negative values of C?

c) What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

69. *The vertical shift D.* Set the constants $A = 3$, $B = 6$, $C = 0$.

a) Plot $f(x)$ for the values $D = 0$, 1, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b) What happens to the graph for negative values of D?

70. *The amplitude A.* Set the constants $B = 6$, $C = D = 0$.

a) Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1$, 5, and 9.

b) What happens to the graph for negative values of A?

PRELIMINARIES — QUESTIONS TO GUIDE YOUR REVIEW

1. What are the order properties of the real numbers? How are they used in solving inequalities?
2. What is a number's absolute value? Give examples. How are $|-a|$, $|ab|$, $|a/b|$, and $|a+b|$ related to $|a|$ and $|b|$?
3. How are absolute values used to describe intervals or unions of intervals? Give examples.
4. How do you find the distance between two points in the coordinate plane?
5. How can you write an equation for a line if you know the coordinates of two points on the line? the line's slope and the coordinates of one point on the line? the line's slope and y-intercept? Give examples.
6. What are the standard equations for lines perpendicular to the coordinate axes?
7. How are the slopes of mutually perpendicular lines related? What about parallel lines? Give examples.
8. When a line is not vertical, what is the relation between its slope and its angle of inclination?
9. What is a function? Give examples. How do you graph a real-valued function of a real variable?
10. Name some typical algebraic and trigonometric functions and draw their graphs.
11. What is an even function? an odd function? What geometric properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd. What, if anything, can you say about sums, products, quotients, and composites involving even and odd functions?
12. If f and g are real-valued functions, how are the domains of $f+g$, $f-g$, fg, and f/g related to the domains of f and g? Give examples.
13. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
14. How do you change the equation $y = f(x)$ to shift its graph up or down? to the left or right? Give examples.
15. Describe the steps you would take to graph the circle $x^2 + y^2 + 4x - 6y + 12 = 0$.
16. If a, b, and c are constants and $a \neq 0$, what can you say about the graph of the equation $y = ax^2 + bx + c$? In particular, how would you go about sketching the curve $y = 2x^2 + 4x$?
17. What inequality describes the points in the coordinate plane that lie inside the circle of radius a centered at the point (h, k)? that lie inside or on the circle? that lie outside the circle? that lie outside or on the circle?
18. What is radian measure? How do you convert from radians to degrees? degrees to radians?
19. Graph the six basic trigonometric functions. What symmetries do the graphs have?
20. How can you sometimes find the values of trigonometric functions from triangles? Give examples.
21. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
22. Starting with the identity $\cos^2\theta + \sin^2\theta = 1$ and the formulas for $\cos(A+B)$ and $\sin(A+B)$, show how a variety of other trigonometric identities may be derived.