a) fg b) f/g c) g/f
d) $f^2 = ff$ e) $g^2 = gg$ f) $f \circ g$
g) $g \circ f$ h) $f \circ f$ i) $g \circ g$

62. Can a function be both even and odd? Give reasons for your answer.

Grapher

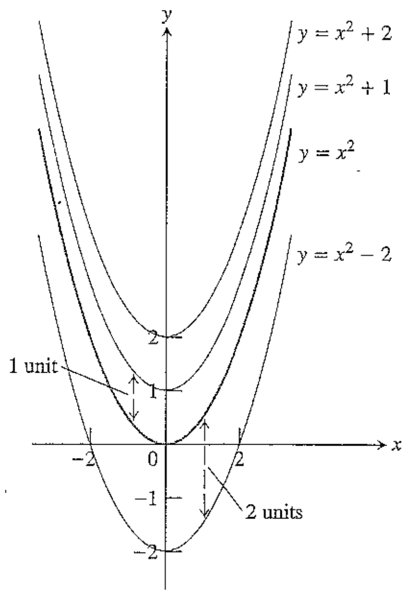
63. (*Continuation of Example 5.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

64. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

4

Shifting Graphs

This section shows how to change an equation to shift its graph up or down or to the right or left. Knowing about this can help us spot familiar graphs in new locations. It can also help us graph unfamiliar equations more quickly. We practice mostly with circles and parabolas (because they make useful examples in calculus), but the methods apply to other curves as well. We will revisit parabolas and circles in Chapter 9.



33 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f.

How to Shift a Graph

To shift the graph of a function $y = f(x)$ straight up, we add a positive constant to the right-hand side of the formula $y = f(x)$.

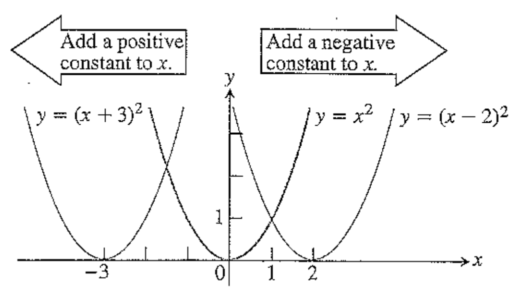
EXAMPLE 1 Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Fig. 33). ❑

To shift the graph of a function $y = f(x)$ straight down, we add a negative constant to the right-hand side of the formula $y = f(x)$.

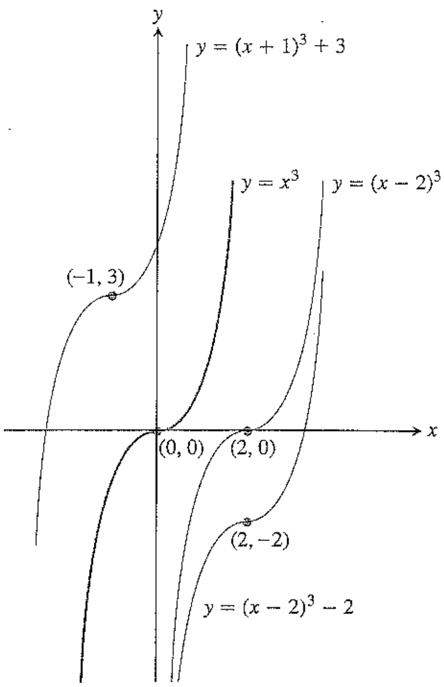
EXAMPLE 2 Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Fig. 33). ❑

To shift the graph of $y = f(x)$ to the left, we add a positive constant to x.

EXAMPLE 3 Adding 3 to x in $y = x^2$ to get $y = (x+3)^2$ shifts the graph 3 units to the left (Fig. 34). ❑



34 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x.



35 The graph of $y = x^3$ shifted to three new positions in the xy-plane.

To shift the graph of $y = f(x)$ to the right, we add a negative constant to x.

EXAMPLE 4 Adding -2 to x in $y = x^2$ to get $y = (x-2)^2$ shifts the graph 2 units to the right (Fig. 34). ☐

Shift Formulas

VERTICAL SHIFTS

$y - k = f(x)$ or	Shifts the graph *up* k units if $k > 0$
$y = f(x) + k$	Shifts it *down* $\|k\|$ units if $k < 0$

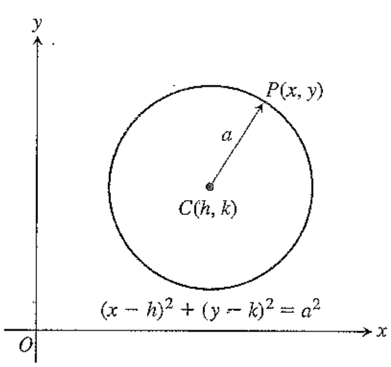
HORIZONTAL SHIFTS

$y = f(x - h)$	Shifts the graph *right* h units *if* $h > 0$
	Shifts it *left* $\|h\|$ units if $h < 0$

EXAMPLE 5 The graph of $y = (x-2)^3 - 2$ is the graph of $y = x^3$ shifted 2 units to the right and 2 units down. The graph of $y = (x+1)^3 + 3$ is the graph of $y = x^3$ shifted 1 unit to the left and 3 units up (Fig. 35). ☐

Equations for Circles

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant (Fig. 36). The fixed point is the **center** of the circle; the constant distance is the **radius**. We saw in Section 2, Example 4, that the circle of radius a centered at the origin has equation $x^2 + y^2 = a^2$. If we shift the circle to place its center at the point (h, k), its equation becomes $(x-h)^2 + (y-k)^2 = a^2$.



36 A circle of radius a in the xy-plane, with center at (h, k).

The Standard Equation for the Circle of Radius a Centered at the Point (h, k)

$$(x-h)^2 + (y-k)^2 = a^2 \tag{1}$$

EXAMPLE 6 If the circle $x^2 + y^2 = 25$ is shifted 2 units to the left and 3 units up, its new equation is $(x+2)^2 + (y-3)^2 = 25$. As Eq. (1) says it should be, this is the equation of the circle of radius 5 centered at $(h, k) = (-2, 3)$. ☐

EXAMPLE 7 The standard equation for the circle of radius 2 centered at (3, 4) is

$$(x-3)^2 + (y-4)^2 = (2)^2$$

or

$$(x-3)^2 + (y-4)^2 = 4.$$

There is no need to square out the x- and y-terms in this equation. In fact, it is better not to do so. The present form reveals the circle's center and radius. ❑

EXAMPLE 8 Find the center and radius of the circle

$$(x-1)^2+(y+5)^2=3.$$

Solution Comparing

$$(x-h)^2+(y-k)^2=a^2$$

with

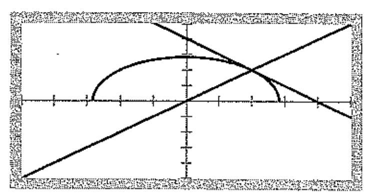
$$(x-1)^2+(y+5)^2=3$$

shows that $h=1$, $k=-5$, and $a=\sqrt{3}$. The center is the point $(h,k)=(1,-5)$; the radius is $a=\sqrt{3}$. ❑

Technology *Square Windows* We use the term "square window" when the units or scalings on both axes are the same. In a square window graphs are true in shape. They are distorted in a nonsquare window.

The term square window does not refer to the shape of the graphic display. Graphing calculators usually have rectangular displays. The displays of Computer Algebra Systems are usually square. When a graph is displayed, the x-unit may differ from the y-unit in order to fit the graph in the display, resulting in a distorted picture. The graphing window can be made square by shrinking or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window "square." If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

On your graphing utility, compare the perpendicular lines $y_1=x$ and $y_2=-x+4$ in a square window and a nonsquare one such as $[-10,10]$ by $[10,10]$. Graph the semicircle $y=\sqrt{8-x^2}$ in the same windows.

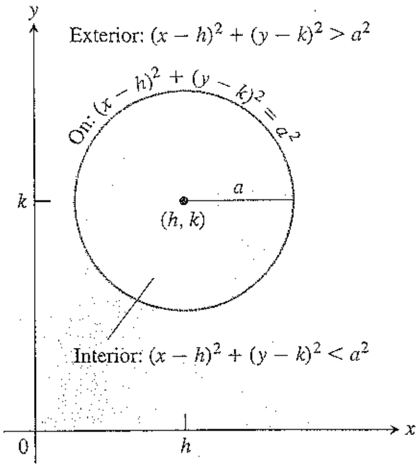


Two perpendicular lines and a semicircle graphed distorted by a rectangular window.

If an equation for a circle is not in standard form, we can find the circle's center and radius by first converting the equation to standard form. The algebraic technique for doing so is *completing the square* (see inside front cover).

EXAMPLE 9 Find the center and radius of the circle

$$x^2+y^2+4x-6y-3=0.$$



37 The interior and exterior of the circle $(x-h)^2+(y-k)^2=a^2$.

Solution We convert the equation to standard form by completing the squares in x and y:

$$x^2+y^2+4x-6y-3=0 \qquad \text{Start with the given equation.}$$

$$(x^2+4x\quad)+(y^2-6y\quad)=3 \qquad \text{Gather terms. Move the constant to the right-hand side.}$$

$$\left(x^2+4x+\left(\frac{4}{2}\right)^2\right)+\left(y^2-6y+\left(\frac{-6}{2}\right)^2\right)= 3+\left(\frac{4}{2}\right)^2+\left(\frac{-6}{2}\right)^2$$

Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2+4x+4)+(y^2-6y+9)=3+4+9$$

$$(x+2)^2+(y-3)^2=16 \qquad \text{Write each quadratic as a squared linear expression.}$$

With the equation now in standard form, we read off the center's coordinates and the radius: $(h, k)=(-2, 3)$ and $a=4$. ❑

Interior and Exterior

The points that lie inside the circle $(x-h)^2+(y-k)^2=a^2$ are the points less than a units from (h, k). They satisfy the inequality

$$(x-h)^2+(y-k)^2<a^2.$$

They make up the region we call the **interior** of the circle (Fig. 37).

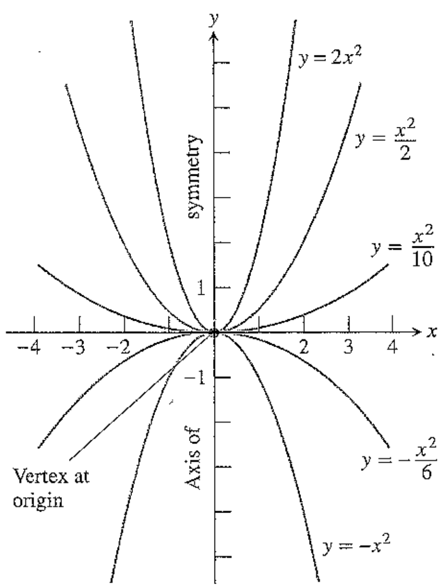
The circle's **exterior** consists of the points that lie more than a units from (h, k). These points satisfy the inequality

$$(x-h)^2+(y-k)^2>a^2.$$

EXAMPLE 10

Inequality	Region
$x^2+y^2<1$	Interior of the unit circle
$x^2+y^2\le 1$	Unit circle plus its interior
$x^2+y^2>1$	Exterior of the unit circle
$x^2+y^2\ge 1$	Unit circle plus its exterior

❑



38 Besides determining the direction in which the parabola $y=ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

Parabolic Graphs

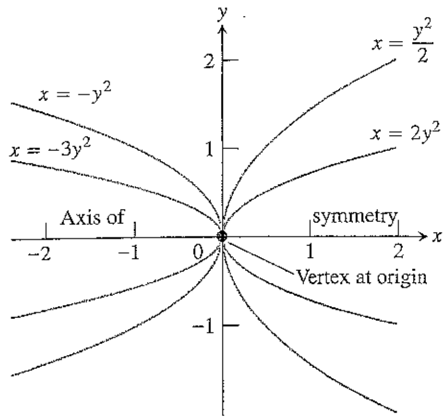
The graph of an equation like $y=3x^2$ or $y=-5x^2$ that has the form

$$y=ax^2$$

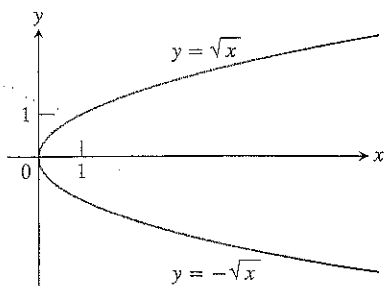
is a **parabola** whose **axis** (axis of symmetry) is the y-axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a>0$ and downward if $a<0$. The larger the value of $|a|$, the narrower the parabola (Fig. 38).

If we interchange x and y in the formula $y=ax^2$, we obtain the equation

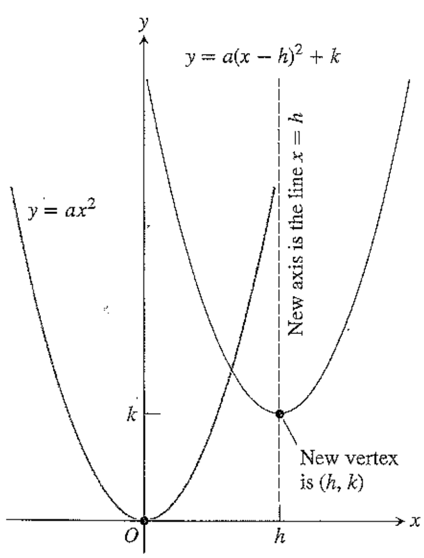
$$x=ay^2.$$



39 The parabola $x = ay^2$ is symmetric about the x-axis. It opens to the right if $a > 0$ and to the left if $a < 0$.



40 The graphs of the functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ join at the origin to make the graph of the equation $x = y^2$ (Example 11).



41 The parabola $y = ax^2$, $a > 0$, shifted h units to the right and k units up.

With x and y now reversed, the graph is a parabola whose axis is the x-axis and whose vertex lies at the origin (Fig. 39).

EXAMPLE 11 The formula $x = y^2$ gives x as a function of y but does *not* give y as a function of x. If we solve for y, we find that $y = \pm\sqrt{x}$. For each positive value of x we get *two* values of y instead of the required single value.

When taken separately, the formulas $y = \sqrt{x}$ and $y = -\sqrt{x}$ do define functions of x. Each formula gives exactly one value of y for each possible value of x. The graph of $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$. The graph of $y = -\sqrt{x}$ is the lower half (Fig. 40). ☐

The Quadratic Equation $y = ax^2 + bx + c, \quad a \neq 0$

To shift the parabola $y = ax^2$ horizontally, we rewrite the equation as

$$y = a(x - h)^2.$$

To shift it vertically as well, we change the equation to

$$y - k = a(x - h)^2. \tag{2}$$

The combined shifts place the vertex at the point (h, k) and the axis along the line $x = h$ (Fig. 41).

Normally there would be no point in multiplying out the right-hand side of Eq. (2). In this case, however, we can learn something from doing so because the resulting equation, when rearranged, takes the form

$$y = ax^2 + bx + c. \tag{3}$$

This tells us that the graph of every equation of the form $y = ax^2 + bx + c$, $a \neq 0$, is the graph of $y = ax^2$ shifted somewhere else. Why? Because the steps that take us from Eq. (2) to Eq. (3) can be reversed to take us from (3) back to (2). The curve $y = ax^2 + bx + c$ has the same shape and orientation as the curve $y = ax^2$.

The axis of the parabola $y = ax^2 + bx + c$ turns out to be the line $x = -b/(2a)$. The y-intercept, $y = c$, is obtained by setting $x = 0$.

The Graph of $y = ax^2 + bx + c, \quad a \neq 0$

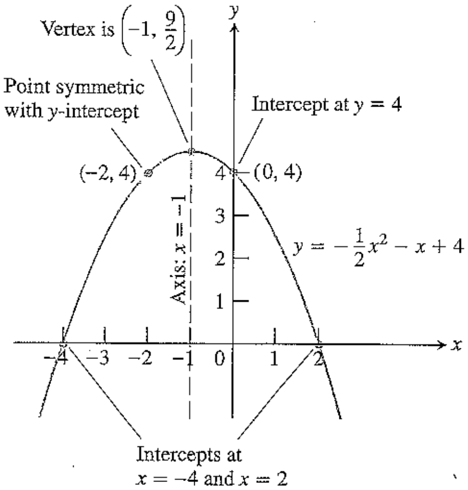
The graph of the equation $y = ax^2 + bx + c$, $a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The axis is the line

$$x = -\frac{b}{2a}. \tag{4}$$

The vertex of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

EXAMPLE 12 *Graphing a parabola*

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.



42 The parabola in Example 12.

Solution We take the following steps.

Step 1: *Compare the equation with $y = ax^2 + bx + c$ to identify a, b, and c.*

$$a = -\frac{1}{2}, \qquad b = -1, \qquad c = 4$$

Step 2: *Find the direction of opening.* Down, because $a < 0$.

Step 3: *Find the axis and vertex.* The axis is the line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1, \qquad \text{Eq. (4)}$$

so the x-coordinate of the vertex is -1. The y-coordinate is

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

Step 4: *Find the x-intercepts (if any).*

$$-\frac{1}{2}x^2 - x + 4 = 0 \qquad \text{Set } y = 0 \text{ in the parabola's equation.}$$

$$x^2 + 2x - 8 = 0 \qquad \text{Solve as usual.}$$

$$(x - 2)(x + 4) = 0$$

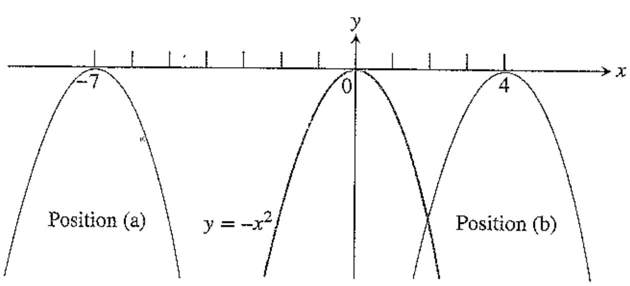
$$x = 2, \qquad x = -4$$

Step 5: *Sketch the graph.* We plot points, sketch the axis (lightly), and use what we know about symmetry and the direction of opening to complete the graph (Fig. 42). ☐

Exercises 4

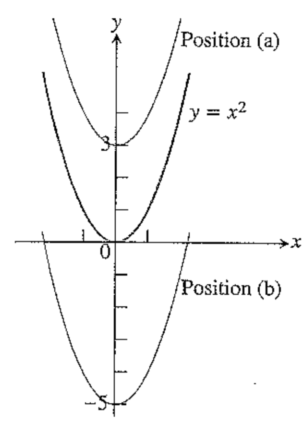
Shifting Graphs

1. Figure 43 shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



43 The parabolas in Exercise 1.

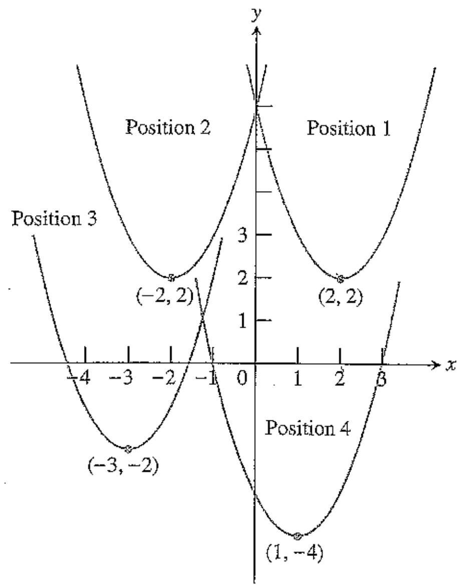
2. Figure 44 shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



44 The parabolas in Exercise 2.

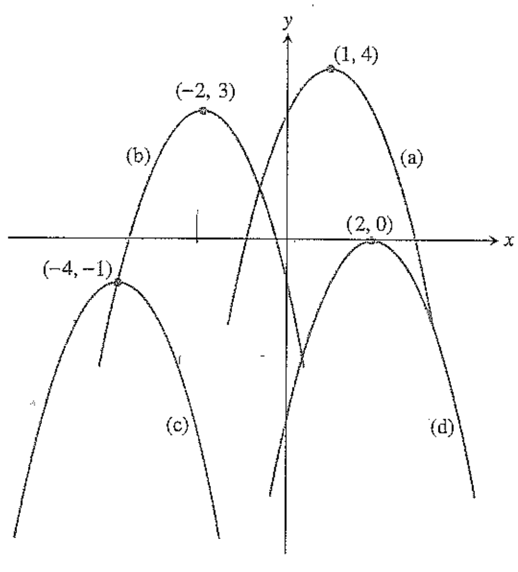
3. Match the equations listed in (a)–(d) to the graphs in Fig. 45.

a) $y=(x-1)^2-4$ b) $y=(x-2)^2+2$
c) $y=(x+2)^2+2$ d) $y=(x+3)^2-2$



45 The parabolas in Exercise 3.

4. Figure 46 shows the graph of $y=-x^2$ shifted to four new positions. Write an equation for each new graph.



46 The parabolas in Exercise 4.

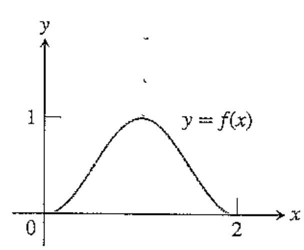
Exercises 5–16 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation. Use the graphs in Fig. 26 for reference as needed.

5. $x^2+y^2=49$ Down 3, left 2
6. $x^2+y^2=25$ Up 3, left 4
7. $y=x^3$ Left 1, down 1
8. $y=x^{2/3}$ Right 1, down 1
9. $y=\sqrt{x}$ Left 0.81
10. $y=-\sqrt{x}$ Right 3
11. $y=2x-7$ Up 7
12. $y=\frac{1}{2}(x+1)+5$ Down 5, right 1
13. $x=y^2$ Left 1
14. $x=-3y^2$ Up 2, right 3
15. $y=1/x$ Up 1, right 1
16. $y=1/x^2$ Left 2, down 1

Graph the functions in Exercises 17–36. Use the graphs in Fig. 26 for reference as needed.

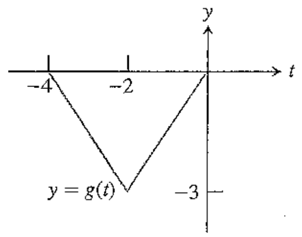
17. $y=\sqrt{x+4}$
18. $y=\sqrt{9-x}$
19. $y=|x-2|$
20. $y=|1-x|-1$
21. $y=1+\sqrt{x-1}$
22. $y=1-\sqrt{x}$
23. $y=(x+1)^{2/3}$
24. $y=(x-8)^{2/3}$
25. $y=1-x^{2/3}$
26. $y+4=x^{2/3}$
27. $y=\sqrt[3]{x-1}-1$
28. $y=(x+2)^{3/2}+1$
29. $y=\dfrac{1}{x-2}$
30. $y=\dfrac{1}{x}-2$
31. $y=\dfrac{1}{x}+2$
32. $y=\dfrac{1}{x+2}$
33. $y=\dfrac{1}{(x-1)^2}$
34. $y=\dfrac{1}{x^2}-1$
35. $y=\dfrac{1}{x^2}+1$
36. $y=\dfrac{1}{(x+1)^2}$

37. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $f(x)+2$ b) $f(x)-1$
c) $2f(x)$ d) $-f(x)$
e) $f(x+2)$ f) $f(x-1)$
g) $f(-x)$ h) $-f(x+1)+1$

38. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $g(-t)$
b) $-g(t)$
c) $g(t)+3$
d) $1-g(t)$
e) $g(-t+2)$
f) $g(t-2)$
g) $g(1-t)$
h) $-g(t-4)$

Circles

In Exercises 39–44, find an equation for the circle with the given center $C(h, k)$ and radius a. Then sketch the circle in the xy-plane. Include the circle's center in your sketch. Also, label the circle's x- and y-intercepts, if any, with their coordinate pairs.

39. $C(0, 2),\quad a=2$
40. $C(-3, 0),\quad a=3$
41. $C(-1, 5),\quad a=\sqrt{10}$
42. $C(1, 1),\quad a=\sqrt{2}$
43. $C(-\sqrt{3}, -2),\quad a=2$
44. $C(3, 1/2),\quad a=5$

Graph the circles whose equations are given in Exercises 45–50. Label each circle's center and intercepts (if any) with their coordinate pairs.

45. $x^2+y^2+4x-4y+4=0$
46. $x^2+y^2-8x+4y+16=0$
47. $x^2+y^2-3y-4=0$
48. $x^2+y^2-4x-(9/4)=0$
49. $x^2+y^2-4x+4y=0$
50. $x^2+y^2+2x=3$

Parabolas

Graph the parabolas in Exercises 51–58. Label the vertex, axis, and intercepts in each case.

51. $y=x^2-2x-3$
52. $y=x^2+4x+3$
53. $y=-x^2+4x$
54. $y=-x^2+4x-5$
55. $y=-x^2-6x-5$
56. $y=2x^2-x+3$
57. $y=\frac{1}{2}x^2+x+4$
58. $y=-\frac{1}{4}x^2+2x+4$

59. Graph the parabola $y=x-x^2$. Then find the domain and range of $f(x)=\sqrt{x-x^2}$.

60. Graph the parabola $y=3-2x-x^2$. Then find the domain and range of $g(x)=\sqrt{3-2x-x^2}$.

Inequalities

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 61–68.

61. $x^2+y^2>7$
62. $x^2+y^2<5$
63. $(x-1)^2+y^2\le 4$
64. $x^2+(y-2)^2\ge 4$
65. $x^2+y^2>1,\quad x^2+y^2<4$
66. $x^2+y^2\le 4,\quad (x+2)^2+y^2\le 4$
67. $x^2+y^2+6y<0,\quad y>-3$
68. $x^2+y^2-4x+2y>4,\quad x>2$

69. Write an inequality that describes the points that lie inside the circle with center $(-2, 1)$ and radius $\sqrt{6}$.

70. Write an inequality that describes the points that lie outside the circle with center $(-4, 2)$ and radius 4.

71. Write a pair of inequalities that describe the points that lie inside or on the circle with center $(0, 0)$ and radius $\sqrt{2}$, and on or to the right of the vertical line through $(1, 0)$.

72. Write a pair of inequalities that describe the points that lie outside the circle with center $(0, 0)$ and radius 2, and inside the circle that has center $(1, 3)$ and passes through the origin.

Shifting Lines

73. The line $y=mx$, which passes through the origin, is shifted vertically and horizontally to pass through the point (x_0, y_0). Find an equation for the new line. (This equation is called the line's *point–slope equation.*)

74. The line $y=mx$ is shifted vertically to pass through the point $(0, b)$. What is the new line's equation?

Intersecting Lines, Circles, and Parabolas

In Exercises 75–82, graph the two equations and find the points in which the graphs intersect.

75. $y=2x,\quad x^2+y^2=1$
76. $x+y=1,\quad (x-1)^2+y^2=1$
77. $y-x=1,\quad y=x^2$
78. $x+y=0,\quad y=-(x-1)^2$
79. $y=-x^2,\quad y=2x^2-1$
80. $y=\frac{1}{4}x^2,\quad y=(x-1)^2$
81. $x^2+y^2=1,\quad (x-1)^2+y^2=1$
82. $x^2+y^2=1,\quad x^2+y=1$

CAS Explorations and Projects

In Exercises 83–86, you will explore graphically what happens to the graph of $y=f(ax)$ as you change the value of the constant a. Use a CAS or computer grapher to perform the following steps.

a) Plot the function $y = f(x)$ together with the function $y = f(ax)$ for $a = 2, 3$, and 10 over the specified interval. Describe what happens to the graph as a increases through positive values.
b) Plot the function $y = f(x)$ and $y = f(ax)$ for the negative values $a = -2, -3$. What happens to the graph in this situation?
c) Plot the function $y = f(x)$ and $y = f(ax)$ for the fractional values $a = 1/2, 1/3, 1/4$. Describe what happens to the graph when $|a| < 1$.

83. $f(x) = \dfrac{5x}{x^2+4}$, $[-10, 10]$

84. $f(x) = \dfrac{2x(x-1)}{x^2+1}$, $[-3, 2]$

85. $f(x) = \dfrac{x+1}{2x^2+1}$, $[-2, 2]$

86. $f(x) = \dfrac{x^4 - 4x^3 + 10}{x^2+4}$, $[-1, 4]$

5

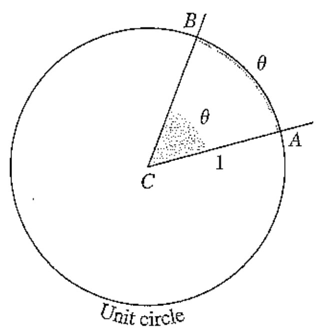
Trigonometric Functions

This section reviews radian measure, trigonometric functions, periodicity, and basic trigonometric identities.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations (Section 2.4).

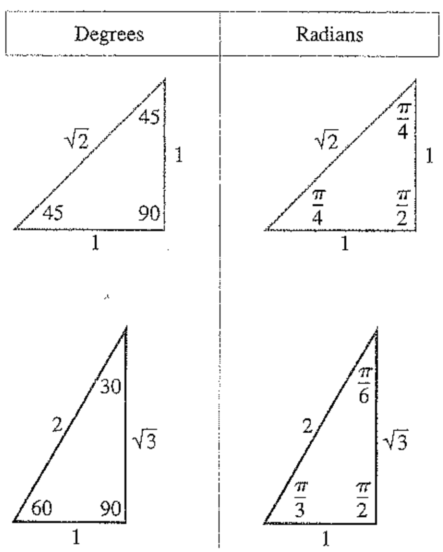
Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle *ACB* is the length of the arc *AB*.

The **radian measure** θ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by the following equation.

$$\pi \text{ radians} = 180°$$



48 The angles of two common triangles, in degrees and radians.

EXAMPLE 1 *Conversions (Fig. 48)*

Convert 45° to radians: $\quad 45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\quad \dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30°$