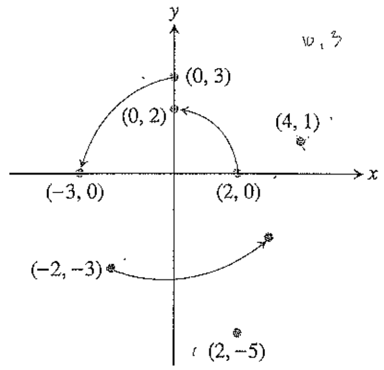
52. A 90° rotation counterclockwise about the origin takes (2, 0) to (0, 2), and (0, 3) to (−3, 0), as shown in Fig. 22. Where does it take each of the following points?

a) (4, 1) **b)** (−2, −3) **c)** (2, −5)
d) $(x, 0)$ **e)** $(0, y)$ **f)** (x, y)
g) What point is taken to (10, 3)?



22 The points moved by the 90° rotation in Exercise 52.

53. For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

54. Find the line that passes through the point (1, 2) and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

55. Show that the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

is the midpoint of the line segment joining $P(x_1, y_1)$ to $Q(x_2, y_2)$.

56. *The distance from a point to a line.* We can find the distance from a point $P(x_0, y_0)$ to a line $L: Ax + By = C$ by taking the following steps (there is a somewhat faster method in Section 10.5):

1. Find an equation for the line M through P perpendicular to L.
2. Find the coordinates of the point Q in which M and L intersect.
3. Find the distance from P to Q.

Use these steps to find the distance from P to L in each of the following cases.

a) $P(2, 1)$, $L: y = x + 2$
b) $P(4, 6)$, $L: 4x + 3y = 12$
c) $P(a, b)$, $L: x = -1$
d) $P(x_0, y_0)$, $L: Ax + By = C$

3 Functions

Functions are the major tools for describing the real world in mathematical terms. This section reviews the notion of function and discusses some of the functions that arise in calculus.

Functions

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x.

The letters used for variable quantities may come from what is being described. When we study circles, we usually call the area A and the radius r. Since $A = \pi r^2$, we say that A is a function of r. The equation $A = \pi r^2$ is a *rule* that tells how to calculate a *unique* (single) output value of A for each possible input value of the radius r.

The set of all possible input values for the radius is called the **domain** of the function. The set of all output values of the area is the **range** of the function. Since circles cannot have negative radii or areas, the domain and range of the circle area function are both the interval $[0, \infty)$, consisting of all nonnegative real numbers.

The domain and range of a mathematical function can be any sets of objects; they do not have to consist of numbers. Most of the domains and ranges we will encounter in this book, however, will be sets of real numbers.

Leonhard Euler (1707–1783)

Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician who ever lived, was also an astronomer, physicist, botanist, chemist, and expert in Oriental languages. He was the first scientist to give the function concept the prominence in his work that it has in mathematics today. Euler's collected books and papers fill 70 volumes. His introductory algebra text, written originally in German (Euler was Swiss), is still read in English translation.

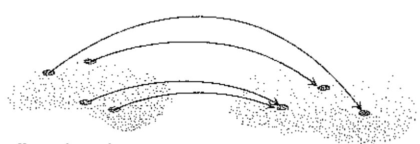
In calculus we often want to refer to a generic function without having any particular formula in mind. Euler invented a symbolic way to say "y is a function of x" by writing

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{''})$$

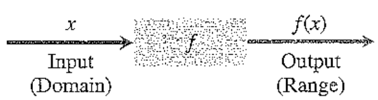
In this notation, the symbol f represents the function. The letter x, called the **independent variable,** represents an input value from the domain of f, and y, the **dependent variable,** represents the corresponding output value $f(x)$ in the range of f. Here is the formal definition of *function.*

Definition

A **function** from a set D to a set R is a rule that assigns a *unique* element $f(x)$ in R to each element x in D.



23 A function from a set D to a set R assigns a unique element of R to each element in D.



24 A "machine" diagram for a function.

In this definition, $D = D(f)$ (read "D of f") is the domain of the function f and R is a set *containing* the range of f. See Fig. 23.

Think of a function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Fig. 24).

In this book we will usually define functions in one of two ways:

1. by giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function, or
2. by giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

Strictly speaking, we should call the function f and not $f(x)$, as the latter denotes the value of the function at the point x. However, as is common usage, we will often refer to the function as $f(x)$ in order to name the variable on which f depends.

It is sometimes convenient to use a single letter to denote both a function and its dependent variable. For instance, we might say that the area A of a circle of radius r is given by the function $A(r) = \pi r^2$.

Evaluation

As we said earlier, most of the functions in this book will be **real-valued functions** of a **real variable,** functions whose domains and ranges are sets of real numbers. We evaluate such functions by substituting particular values from the domain into the function's defining rule to calculate the corresponding values in the range.

EXAMPLE 1 The volume V of a ball (solid sphere) of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3.$$

The volume of a ball of radius 3 m is

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \text{ m}^3.$$

❑

EXAMPLE 2 Suppose that the function F is defined for all real numbers t by the formula

$$F(t) = 2(t-1) + 3.$$

Evaluate F at the input values 0, 2, $x+2$, and $F(2)$.

Solution In each case we substitute the given input value for t into the formula for F:

$$\begin{aligned} F(0) &= 2(0-1)+3 = -2+3 = 1 \\ F(2) &= 2(2-1)+3 = 2+3 = 5 \\ F(x+2) &= 2(x+2-1)+3 = 2x+5 \\ F(F(2)) &= F(5) = 2(5-1)+3 = 11. \end{aligned}$$

The Domain Convention

When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly, the domain is assumed to be the largest set of x-values for which the formula gives real y-values. This is the function's so-called **natural domain.** If we want the domain to be restricted in some way, we must say so.

The domain of the function $y = x^2$ is understood to be the entire set of real numbers. The formula gives a real y-value for every real number x. If we want to restrict the domain to values of x greater than or equal to 2, we must write "$y = x^2$, $x \geq 2$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2$, $x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In symbols, the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

Most of the functions we encounter will have domains that are either intervals or unions of intervals.

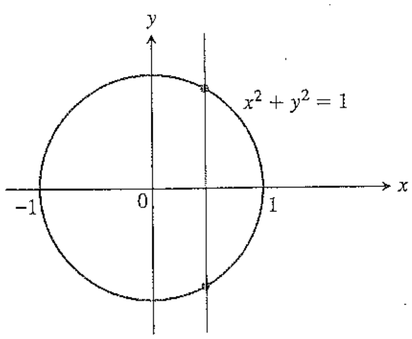
EXAMPLE 3

Function	Domain (x)	Range (y)
$y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$
$y = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4-x}$	$(-\infty, 4]$	$[0, \infty)$

The formula $y = \sqrt{1-x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Beyond this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1-x^2}$ is $[0, 1]$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero.* The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is precisely the set of all nonzero real numbers.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).



25 This circle is not the graph of a function $y = f(x)$; it fails the vertical line test.

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all square roots of nonnegative numbers. ❑

Graphs of Functions

The **graph** of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane whose coordinates (x, y) are input–output pairs for f.

Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no *vertical line* can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Fig. 25). If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.

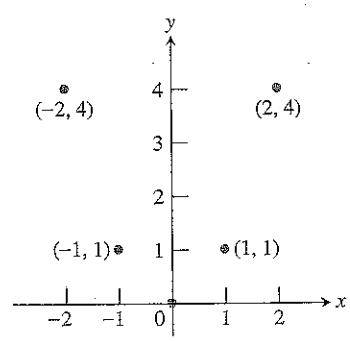
EXAMPLE 4 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution

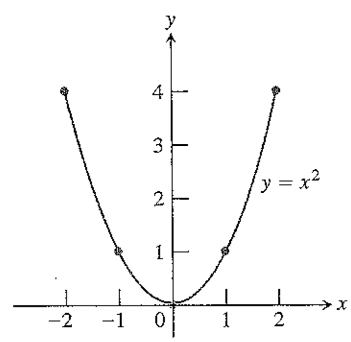
Step 1: Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

x	$y = x^2$
−2	4
−1	1
0	0
1	1
2	4

Step 2: Plot the points (x, y) whose coordinates appear in the table.



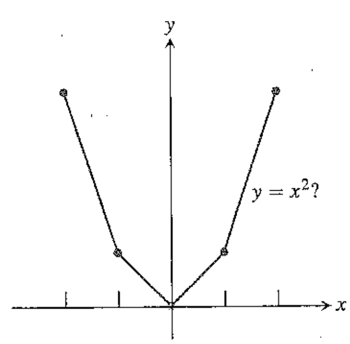
Step 3: Draw a smooth curve through the plotted points. Label the curve with its equation.

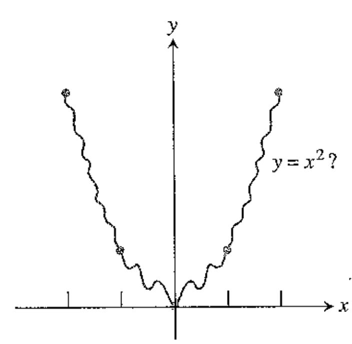


❑

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?

Computers and graphing calculators graph functions in much this way—by stringing together plotted points—and the same question arises.

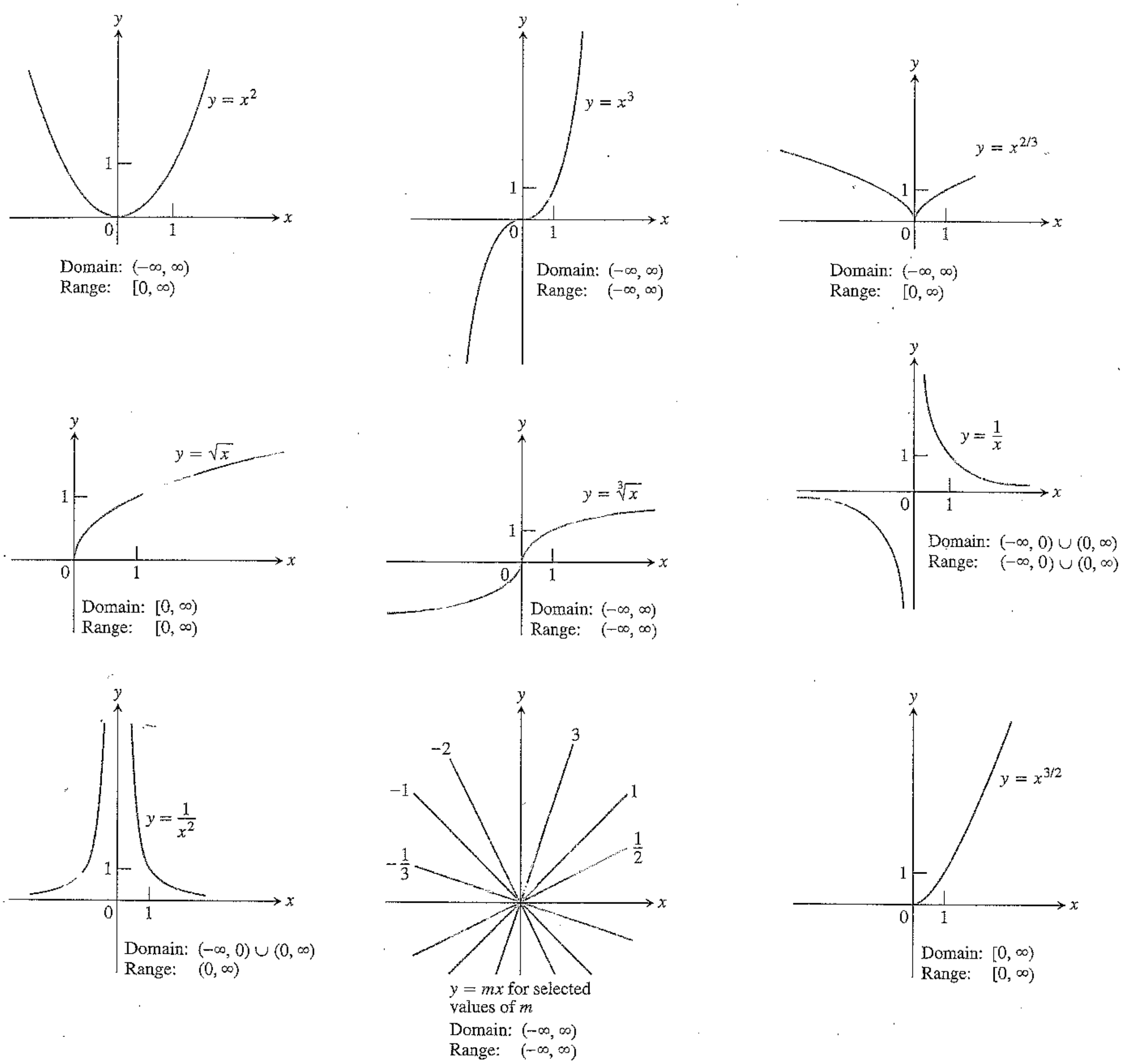




To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? The answer lies in calculus, as we will see in Chapter 3. There we will use a marvelous mathematical tool called the *derivative* to find a curve's shape between plotted points. Meanwhile we will have to settle for plotting points and connecting them as best we can.

Figure 26 shows the graphs of several functions frequently encountered in calculus. It is a good idea to learn the shapes of these graphs so that you can recognize them or sketch them when the need arises.

26 Useful graphs.



Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g, we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x).$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{(where } g(x) \neq 0\text{).}$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 5

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = \sqrt{1 - x}$	$(-\infty, 1]$
$3g$	$3g(x) = 3\sqrt{1 - x}$	$(-\infty, 1]$
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Composite Functions

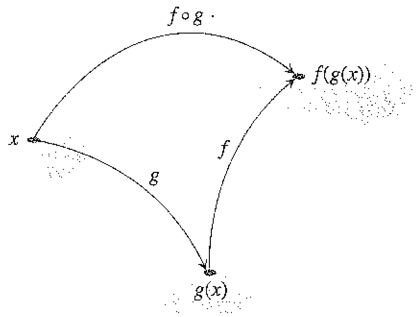
Composition is another method for combining functions.

Definition

If f and g are functions, the **composite** function $f \circ g$ ("f circle g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.



27 The relation of $f \circ g$ to g and f.

The definition says that two functions can be composed when the range of the first lies in the domain of the second (Fig. 27). To find $(f \circ g)(x)$, we *first find* $g(x)$ and *second find* $f(g(x))$.

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 6 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

a) $(f \circ g)(x)$ **b)** $(g \circ f)(x)$ **c)** $(f \circ f)(x)$ **d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$\mathbb{R}$ or $(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, if $x \geq -1$. ❑

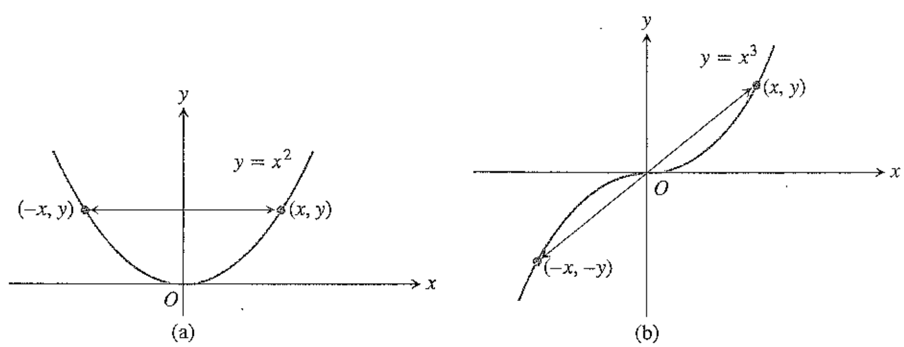
Even Functions and Odd Functions—Symmetry

A function $y = f(x)$ is **even** if $f(-x) = f(x)$ for every number x in the domain of f. Notice that this implies that both x and $-x$ must be in the domain of f. The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$.

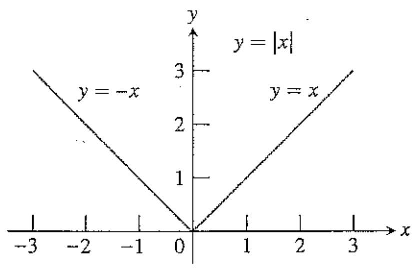
The graph of an even function $y = f(x)$ is symmetric about the y-axis. Since $f(-x) = f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Fig. 28a). Once we know the graph on one side of the y-axis, we automatically know it on the other side.

A function $y = f(x)$ is **odd** if $f(-x) = -f(x)$ for every number x in the domain of f. Again, both x and $-x$ must lie in the domain of f. The function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$.

The graph of an odd function $y = f(x)$ is symmetric about the origin. Since $f(-x) = -f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Fig. 28b). Here again, once we know the graph of f on one side of the y-axis, we know it on both sides.



28 (a) Symmetry about the y-axis. If (x, y) is on the graph, so is $(-x, y)$. (b) Symmetry about the origin. If (x, y) is on the graph, so is $(-x, -y)$.



29 The absolute value function.

Piecewise Defined Functions

Sometimes a function uses different formulas on different parts of its domain. One example is the absolute value function

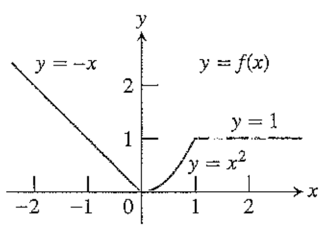
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Fig. 29. Here are some examples.

EXAMPLE 7 The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x (Fig. 30). ❑



30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 7).

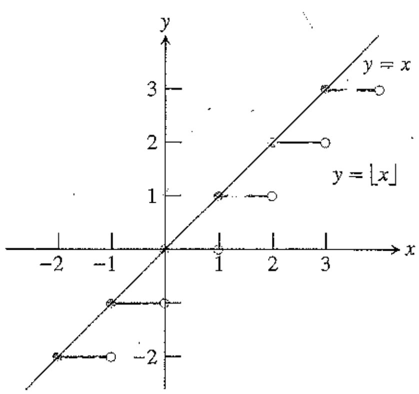
EXAMPLE 8 *The greatest integer function*

The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$. Figure 31 shows the graph. Observe that

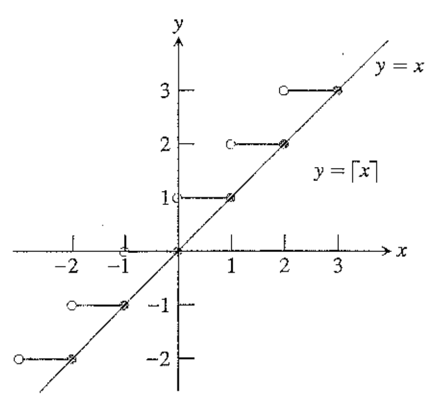
$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$ ❑

EXAMPLE 9 *The least integer function*

The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 32 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges \$1 for each hour or part of an hour.



31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x.



32 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x. ❑

Exercises 3

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(t) = \dfrac{1}{\sqrt{t}}$

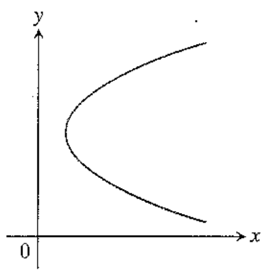
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$

5. $g(z) = \sqrt{4 - z^2}$

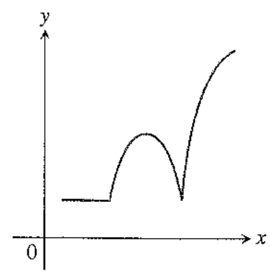
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

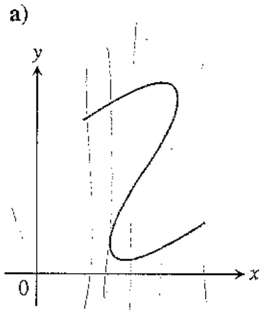
7. a)



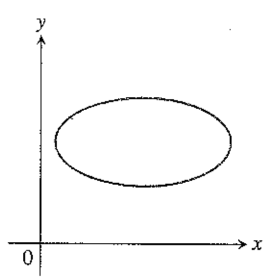
b)



8. a)



b)



Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.

10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.

11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

Functions and Graphs

Graph the functions in Exercises 13–24. What symmetries, if any, do the graphs have? Use the graphs in Fig. 26 for guidance, as needed.

13. $y = -x^3$

14. $y = -\dfrac{1}{x^2}$

15. $y = -\dfrac{1}{x}$

16. $y = \dfrac{1}{|x|}$

17. $y = \sqrt{|x|}$

18. $y = \sqrt{-x}$

19. $y = x^3/8$

20. $y = -4\sqrt{x}$

21. $y = -x^{3/2}$

22. $y = (-x)^{3/2}$

23. $y = (-x)^{2/3}$

24. $y = -x^{2/3}$

25. Graph the following equations and explain why they are not graphs of functions of x.

a) $|y| = x$ b) $y^2 = x^2$

26. Graph the following equations and explain why they are not graphs of functions of x.

a) $|x| + |y| = 1$ b) $|x + y| = 1$

Even and Odd Functions

In Exercises 27–38, say whether the function is even, odd, or neither.

27. $f(x) = 3$

28. $f(x) = x^{-5}$

29. $f(x) = x^2 + 1$

30. $f(x) = x^2 + x$

31. $g(x) = x^3 + x$

32. $g(x) = x^4 + 3x^2 - 1$

33. $g(x) = \dfrac{1}{x^2 - 1}$

34. $g(x) = \dfrac{x}{x^2 - 1}$

35. $h(t) = \dfrac{1}{t - 1}$

36. $h(t) = |t^3|$

37. $h(t) = 2t + 1$

38. $h(t) = 2|t| + 1$

Sums, Differences, Products, and Quotients

In Exercises 39 and 40, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

39. $f(x) = x, \quad g(x) = \sqrt{x - 1}$

40. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 41 and 42, find the domains and ranges of f, g, f/g, and g/f.

41. $f(x) = 2, \quad g(x) = x^2 + 1$

42. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

43. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a) $f(g(0))$ b) $g(f(0))$
c) $f(g(x))$ d) $g(f(x))$
e) $f(f(-5))$ f) $g(g(2))$
g) $f(f(x))$ h) $g(g(x))$

44. If $f(x) = x - 1$ and $g(x) = 1/(x+1)$, find the following.

a) $f(g(1/2))$ b) $g(f(1/2))$
c) $f(g(x))$ d) $g(f(x))$
e) $f(f(2))$ f) $g(g(2))$
g) $f(f(x))$ h) $g(g(x))$

45. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a) $u(v(f(x)))$ b) $u(f(v(x)))$
c) $v(u(f(x)))$ d) $v(f(u(x)))$
e) $f(u(v(x)))$ f) $f(v(u(x)))$

46. If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

a) $h(g(f(x)))$ b) $h(f(g(x)))$
c) $g(h(f(x)))$ d) $g(f(h(x)))$
e) $f(g(h(x)))$ f) $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 47 and 48 as a composite involving one or more of f, g, h, and j.

47. a) $y = \sqrt{x} - 3$ b) $y = 2\sqrt{x}$
c) $y = x^{1/4}$ d) $y = 4x$
e) $y = \sqrt{(x-3)^3}$ f) $y = (2x - 6)^3$

48. a) $y = 2x - 3$ b) $y = x^{3/2}$
c) $y = x^9$ d) $y = x - 6$
e) $y = 2\sqrt{x - 3}$ f) $y = \sqrt{x^3 - 3}$

49. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a)	$x - 7$	$\sqrt{x}$	
b)	$x + 2$	$3x$	
c)		$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d)	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	
e)		$1 + \dfrac{1}{x}$	x
f)	$\dfrac{1}{x}$		x

50. *A magic trick.* You may have heard of a magic trick that goes like this: Take any number. Add 5. Double the result. Subtract 6. Divide by 2. Subtract 2. Now tell me your answer, and I'll tell you what you started with.

Pick a number and try it.

You can see what is going on if you let x be your original number and follow the steps to make a formula $f(x)$ for the number you end up with.

Piecewise Defined Functions

Graph the functions in Exercises 51–54.

51. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

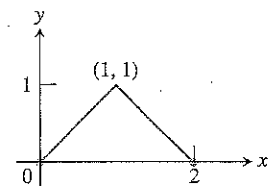
52. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

53. $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

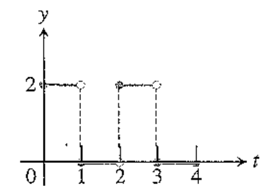
54. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

55. Find a formula for each function graphed.

a)

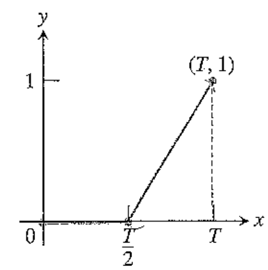


b)

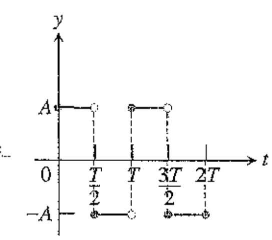


56. Find a formula for each function graphed.

a)



b)



The Greatest and Least Integer Functions

57. For what values of x is (a) $\lfloor x \rfloor = 0$? (b) $\lceil x \rceil = 0$?

58. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

59. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

60. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0 \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Even and Odd Functions

61. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

a) fg b) f/g c) g/f
d) $f^2 = ff$ e) $g^2 = gg$ f) $f \circ g$
g) $g \circ f$ h) $f \circ f$ i) $g \circ g$

62. Can a function be both even and odd? Give reasons for your answer.

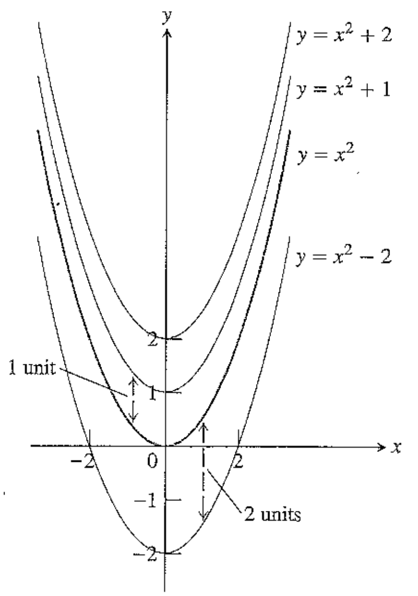
Grapher

63. (*Continuation of Example 5.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

64. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

4 Shifting Graphs

This section shows how to change an equation to shift its graph up or down or to the right or left. Knowing about this can help us spot familiar graphs in new locations. It can also help us graph unfamiliar equations more quickly. We practice mostly with circles and parabolas (because they make useful examples in calculus), but the methods apply to other curves as well. We will revisit parabolas and circles in Chapter 9.



33 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f.

How to Shift a Graph

To shift the graph of a function $y = f(x)$ straight up, we add a positive constant to the right-hand side of the formula $y = f(x)$.

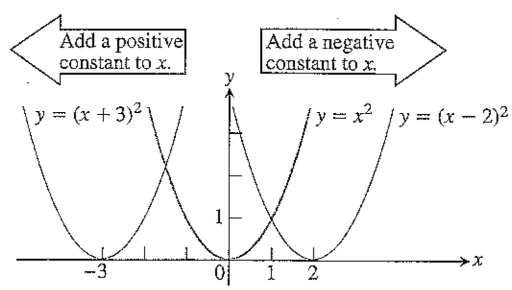
EXAMPLE 1 Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Fig. 33). ❑

To shift the graph of a function $y = f(x)$ straight down, we add a negative constant to the right-hand side of the formula $y = f(x)$.

EXAMPLE 2 Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Fig. 33). ❑

To shift the graph of $y = f(x)$ to the left, we add a positive constant to x.

EXAMPLE 3 Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Fig. 34). ❑



34 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x.