Preliminaries

Overview This chapter reviews the main things you need to know to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, functions, and trigonometry.

1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers and the Real Line

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

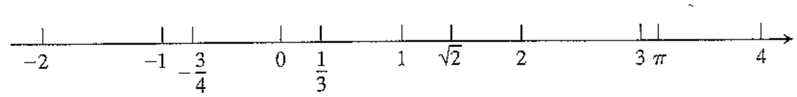
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever.

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol $\mathbb{R}$ donotes either the real number system or, equivalently, the real line.

Properties of Real Numbers

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The algebraic properties say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

The order properties of real numbers are summarized in the following list.

The symbol $\Rightarrow$ means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign.

The completeness property of the real number system is deeper and harder to define precisely. Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many of the theorems of calculus would fail if the real number system were not complete, and the nature of the connection is important. The topic is best saved for a more advanced course, however, and we will not pursue it.

Subsets of $\mathbb{R}$

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

a) terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

b) repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}.$$

The bar indicates the block of repeating digits.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$.

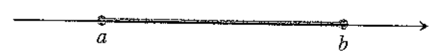
Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \leq x \leq 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together make up what is called the interval's **interior**.

Table 1 Types of intervals

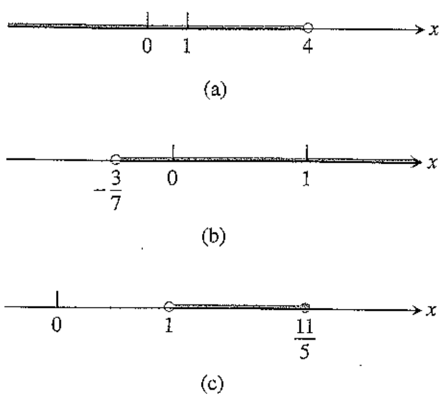
	Notation	Set	Graph
Finite:	(a, b)	$\{x \mid a < x < b\}$	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	
	$[a, b)$	$\{x \mid a \leq x < b\}$	
	$(a, b]$	$\{x \mid a < x \leq b\}$	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	
	$[a, \infty)$	$\{x \mid x \geq a\}$	
	$(-\infty, b)$	$\{x \mid x < b\}$	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

EXAMPLE 1 Solve the following inequalities and graph their solution sets on the real line.

a) $2x - 1 < x + 3$ **b)** $-\dfrac{x}{3} < 2x + 1$ **c)** $\dfrac{6}{x-1} \geq 5$



1 Solutions for Example 1.

Solution

a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \qquad \text{Add 1 to both sides.}$$

$$x < 4 \qquad \text{Subtract } x \text{ from both sides.}$$

The solution set is the interval $(-\infty, 4)$ (Fig. 1a).

b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \qquad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \qquad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \qquad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \qquad \text{Divide by 7.}$$

The solution set is the interval $(-3/7, \infty)$ (Fig. 1b).

c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval $(1, 11/5]$ (Fig. 1c). ❑

Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 $|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$, $\big|-|a|\big| = |a|$ ❑

Notice that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$.

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, an alternate definition of $|x|$ is

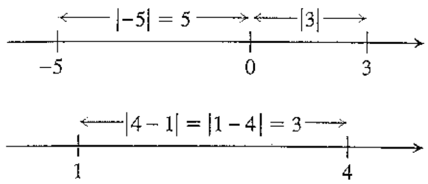
$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

Geometrically, $|x|$ represents the distance from x to the origin 0 on the real line. More generally (Fig. 2)

$$|x - y| = \text{the distance between } x \text{ and } y.$$

The absolute value has the following properties.



2 Absolute values give distances between points on the number line.

Absolute Value Properties

1. $\lvert -a\rvert = \lvert a\rvert$	A number and its negative have the same absolute value.
2. $\lvert ab\rvert = \lvert a\rvert\lvert b\rvert$	The absolute value of a product is the product of the absolute values.
3. $\left\lvert \dfrac{a}{b}\right\rvert = \dfrac{\lvert a\rvert}{\lvert b\rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4. $\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$	**The triangle inequality** The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$.

Notice that absolute value bars in expressions like $|-3+5|$ also work like parentheses: We do the arithmetic inside *before* taking the absolute value.

EXAMPLE 3

$$\begin{aligned} |-3+5| &= |2| = 2 < |-3| + |5| = 8 \\ |3+5| &= |8| = |3| + |5| \\ |-3-5| &= |-8| = 8 = |-3| + |-5| \end{aligned}$$

❑

EXAMPLE 4 Solve the equation $|2x - 3| = 7$.

Solution The equation says that $2x - 3 = \pm 7$, so there are two possibilities:

$$\begin{aligned} 2x - 3 &= 7 & 2x - 3 &= -7 & &\text{Equivalent equations without absolute values} \\ 2x &= 10 & 2x &= -4 & &\text{Solve as usual.} \\ x &= 5 & x &= -2 & & \end{aligned}$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ❑

Inequalities Involving Absolute Values

The inequality $|a| < D$ says that the distance from a to 0 is less than D. Therefore, a must lie between D and $-D$.

The symbol $\Leftrightarrow$ means "if and only if," or "implies and is implied by."

Intervals and Absolute Values

If D is any positive number, then

$$|a| < D \quad \Leftrightarrow \quad -D < a < D, \tag{1}$$

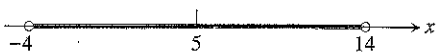
$$|a| \leq D \quad \Leftrightarrow \quad -D \leq a \leq D. \tag{2}$$

EXAMPLE 5 Solve the inequality $|x-5| < 9$ and graph the solution set on the real line.

Solution

$$\begin{aligned} |x-5| &< 9 \\ -9 < x-5 &< 9 && \text{Eq. (1)} \\ -9+5 < x &< 9+5 && \text{Add 5 to each part to isolate } x. \\ -4 < x &< 14 \end{aligned}$$

The solution set is the open interval $(-4, 14)$ (Fig. 3). ❑



3 The solution set of the inequality $|x-5| < 9$ is the interval $(-4, 14)$ graphed here (Example 5).

EXAMPLE 6 Solve the inequality $\left|5 - \dfrac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5 - \frac{2}{x}\right| < 1 \quad &\Leftrightarrow \quad -1 < 5 - \frac{2}{x} < 1 && \text{Eq. (1)} \\ &\Leftrightarrow \quad -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow \quad 3 > \frac{1}{x} > 2 && \text{Multiply by } -\frac{1}{2}. \\ &\Leftrightarrow \quad \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$. ❑

EXAMPLE 7 Solve the inequality and graph the solution set:

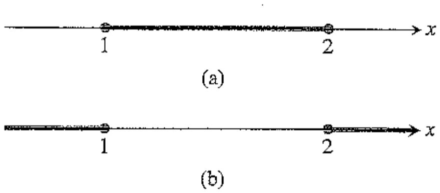
a) $|2x-3| \leq 1$ **b)** $|2x-3| \geq 1$

Solution

a)

$$\begin{aligned} |2x-3| &\leq 1 \\ -1 \leq 2x-3 &\leq 1 && \text{Eq. (2)} \\ 2 \leq 2x &\leq 4 && \text{Add 3.} \\ 1 \leq x &\leq 2 && \text{Divide by 2.} \end{aligned}$$

The solution set is the closed interval $[1, 2]$ (Fig. 4a).



4 Graphs of the solution sets (a) $[1, 2]$ and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 7.

Union and intersection

Notice the use of the symbol $\cup$ to denote the union of intervals. A number lies in the **union** of two sets if it lies in either set. Similarly we use the symbol $\cap$ to denote intersection. A number lies in the **intersection** $I \cap J$ of two sets if it lies in *both* sets I and J. For example, $[1, 3) \cap [2, 4] = [2, 3)$.

b)

$$|2x - 3| \geq 1$$

$2x - 3 \geq 1$	or	$-(2x - 3) \geq 1$	
$2x - 3 \geq 1$	or	$2x - 3 \leq -1$	Multiply second inequality by -1.
$x - \frac{3}{2} \geq \frac{1}{2}$	or	$x - \frac{3}{2} \leq -\frac{1}{2}$	Divide by 2.
$x \geq 2$	or	$x \leq 1$	Add $\frac{3}{2}$.

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Fig. 4b). ❑

Exercises 1

Decimal Representations

1. Express $1/9$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/9$? $3/9$? $8/9$?

2. Express $1/11$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/11$? $3/11$? $9/11$?

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

   a) $0 < x < 4$ b) $0 < x - 2 < 4$
   c) $1 < \frac{x}{2} < 3$ d) $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$
   e) $1 < \frac{6}{x} < 3$ f) $|x - 4| < 2$
   g) $-6 < -x < 2$ h) $-6 < -x < -2$

4. If $-1 < y - 5 < 1$, which of the following statements about y are necessarily true, and which are not necessarily true?

   a) $4 < y < 6$ b) $-6 < y < -4$
   c) $y > 4$ d) $y < 6$
   e) $0 < y - 4 < 2$ f) $2 < \frac{y}{2} < 3$
   g) $\frac{1}{6} < \frac{1}{y} < \frac{1}{4}$ h) $|y - 5| < 1$

In Exercises 5–12, solve the inequalities and graph the solution sets.

5. $-2x > 4$
6. $8 - 3x \geq 5$
7. $5x - 3 \leq 7 - 3x$
8. $3(2 - x) > 2(3 + x)$
9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$
10. $\frac{6 - x}{4} < \frac{3x - 4}{2}$
11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$
12. $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$
14. $|y - 3| = 7$
15. $|2t + 5| = 4$
16. $|1 - t| = 1$
17. $|8 - 3s| = \frac{9}{2}$
18. $\left|\frac{s}{2} - 1\right| = 1$

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, graph each solution set on the real line.

19. $|x| < 2$
20. $|x| \leq 2$
21. $|t - 1| \leq 3$
22. $|t + 2| < 1$
23. $|3y - 7| < 4$
24. $|2y + 5| < 1$
25. $\left|\frac{z}{5} - 1\right| \leq 1$
26. $\left|\frac{3}{2}z - 1\right| \leq 2$
27. $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$
28. $\left|\frac{2}{x} - 4\right| < 3$
29. $|2s| \geq 4$
30. $|s + 3| \geq \frac{1}{2}$
31. $|1 - x| > 1$
32. $|2 - 3x| > 5$
33. $\left|\frac{r + 1}{2}\right| \geq 1$
34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Quadratic Inequalities

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and graph them. Use the result $\sqrt{a^2} = |a|$ as appropriate.

35. $x^2 < 2$
36. $4 \leq x^2$
37. $4 < x^2 < 9$
38. $\frac{1}{9} < x^2 < \frac{1}{4}$
39. $(x - 1)^2 < 4$
40. $(x + 3)^2 < 2$
41. $x^2 - x < 0$
42. $x^2 - x - 2 \geq 0$

Theory and Examples

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?

44. Solve the equation $|x-1| = 1-x$.

45. *A proof of the triangle inequality.* Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$
\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2 \\
&\le a^2+2|a||b|+b^2 && (2)\\
&\le |a|^2+2|a||b|+|b|^2 && (3)\\
&= (|a|+|b|)^2 \\
|a+b| &\le |a|+|b| && (4)
\end{aligned}
$$

46. Prove that $|ab| = |a||b|$ for any numbers a and b.

47. If $|x| \le 3$ and $x > -1/2$, what can you say about x?

48. Graph the inequality $|x|+|y| \le 1$.

49. GRAPHER

a) Graph the functions $f(x) = x/2$ and $g(x) = 1+(4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b) Confirm your findings in (a) algebraically.

50. GRAPHER

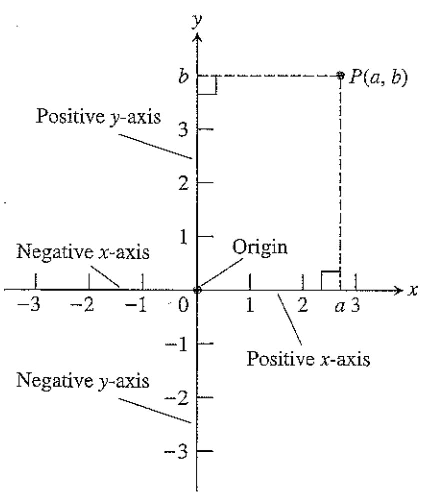
a) Graph the functions $f(x) = 3/(x-1)$ and $g(x) = 2/(x+1)$ together to identify the values of x for which

$$\frac{3}{x-1} < \frac{2}{x+1}.$$

b) Confirm your findings in (a) algebraically.

2 Coordinates, Lines, and Increments

This section reviews coordinates and lines and discusses the notion of increment.



5 Cartesian coordinates.

Cartesian Coordinates in the Plane

The positions of all points in the plane can be measured with respect to two perpendicular real lines in the plane intersecting in the 0-point of each (Fig. 5). These lines are called **coordinate axes** in the plane. On the horizontal x-axis, numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upward. The point where x and y are both 0 is the **origin** of the coordinate system, often denoted by the letter O.

If P is any point in the plane, we can draw lines through P perpendicular to the two coordinate axes. If the lines meet the x-axis at a and the y-axis at b, then a is the ***x*-coordinate** of P, and b is the ***y*-coordinate**. The ordered pair (a, b) is the point's **coordinate pair**. The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point $(0, 0)$.

The origin divides the x-axis into the **positive *x*-axis** to the right and the **negative *x*-axis** to the left. It divides the y-axis into the **positive** and **negative *y*-axis** above and below. The axes divide the plane into four regions called **quadrants**, numbered counterclockwise as in Fig. 6.

A Word About Scales

When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If we plot time vs. thrust for a rocket motor, for example, there is no reason to place the mark that shows 1 sec on the time axis the same distance from the origin as the mark that shows 1 lb on the thrust axis.

When we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry and trigonometry, we try to make the scales on the axes identical. A vertical unit