$-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b) What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

68. *The horizontal shift C.* Set the constants $A = 3$, $B = 6$, $D = 0$.

a) Plot $f(x)$ for the values $C = 0$, 1, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b) What happens to the graph for negative values of C?

c) What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

69. *The vertical shift D.* Set the constants $A = 3$, $B = 6$, $C = 0$.

a) Plot $f(x)$ for the values $D = 0$, 1, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b) What happens to the graph for negative values of D?

70. *The amplitude A.* Set the constants $B = 6$, $C = D = 0$.

a) Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1$, 5, and 9.

b) What happens to the graph for negative values of A?

PRELIMINARIES — QUESTIONS TO GUIDE YOUR REVIEW

1. What are the order properties of the real numbers? How are they used in solving inequalities?
2. What is a number's absolute value? Give examples. How are $|-a|$, $|ab|$, $|a/b|$, and $|a+b|$ related to $|a|$ and $|b|$?
3. How are absolute values used to describe intervals or unions of intervals? Give examples.
4. How do you find the distance between two points in the coordinate plane?
5. How can you write an equation for a line if you know the coordinates of two points on the line? the line's slope and the coordinates of one point on the line? the line's slope and y-intercept? Give examples.
6. What are the standard equations for lines perpendicular to the coordinate axes?
7. How are the slopes of mutually perpendicular lines related? What about parallel lines? Give examples.
8. When a line is not vertical, what is the relation between its slope and its angle of inclination?
9. What is a function? Give examples. How do you graph a real-valued function of a real variable?
10. Name some typical algebraic and trigonometric functions and draw their graphs.
11. What is an even function? an odd function? What geometric properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd. What, if anything, can you say about sums, products, quotients, and composites involving even and odd functions?
12. If f and g are real-valued functions, how are the domains of $f+g$, $f-g$, fg, and f/g related to the domains of f and g? Give examples.
13. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
14. How do you change the equation $y = f(x)$ to shift its graph up or down? to the left or right? Give examples.
15. Describe the steps you would take to graph the circle $x^2 + y^2 + 4x - 6y + 12 = 0$.
16. If a, b, and c are constants and $a \ne 0$, what can you say about the graph of the equation $y = ax^2 + bx + c$? In particular, how would you go about sketching the curve $y = 2x^2 + 4x$?
17. What inequality describes the points in the coordinate plane that lie inside the circle of radius a centered at the point (h, k)? that lie inside or on the circle? that lie outside the circle? that lie outside or on the circle?
18. What is radian measure? How do you convert from radians to degrees? degrees to radians?
19. Graph the six basic trigonometric functions. What symmetries do the graphs have?
20. How can you sometimes find the values of trigonometric functions from triangles? Give examples.
21. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
22. Starting with the identity $\cos^2\theta + \sin^2\theta = 1$ and the formulas for $\cos(A+B)$ and $\sin(A+B)$, show how a variety of other trigonometric identities may be derived.

PRELIMINARIES PRACTICE EXERCISES

Geometry

1. A particle in the plane moved from $A(-2, 5)$ to the y-axis in such a way that Δy equaled $3\,\Delta x$. What were the particle's new coordinates?

2. a) Plot the points $A(8, 1)$, $B(2, 10)$, $C(-4, 6)$, $D(2, -3)$, and $E(14/3, 6)$.
   b) Find the slopes of the lines AB, BC, CD, DA, CE, and BD.
   c) Do any four of the five points A, B, C, D, and E form a parallelogram?
   d) Are any three of the five points collinear? How do you know?
   e) Which of the lines determined by the five points pass through the origin?

3. Do the points $A(6, 4)$, $B(4, -3)$, and $C(-2, 3)$ form an isosceles triangle? a right triangle? How do you know?

4. Find the coordinates of the point on the line $y = 3x + 1$ that is equidistant from $(0, 0)$ and $(-3, 4)$.

Functions and Graphs

5. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.

6. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.

7. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.

8. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of lift-off. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

Composition with absolute values. In Exercises 9–14, graph f_1 and f_2 together. Then describe how applying the absolute value function before applying f_1 affects the graph.

	$f_1(x)$	$f_2(x) = f_1(\lvert x\rvert)$
9.	x	$\lvert x\rvert$
10.	x^3	$\lvert x\rvert^3$
11.	x^2	$\lvert x\rvert^2$
12.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x\rvert}$
13.	$\sqrt{x}$	$\sqrt{\lvert x\rvert}$
14.	$\sin x$	$\sin\lvert x\rvert$

Composition with absolute values. In Exercises 15–20, graph g_1 and g_2 together. Then describe how taking absolute values after applying g_1 affects the graph.

	$g_1(x)$	$g_2(x) = \lvert g_1(x)\rvert$
15.	x^3	$\lvert x^3\rvert$
16.	$\sqrt{x}$	$\lvert\sqrt{x}\rvert$
17.	$\dfrac{1}{x}$	$\left\lvert\dfrac{1}{x}\right\rvert$
18.	$4 - x^2$	$\lvert 4 - x^2\rvert$
19.	$x^2 + x$	$\lvert x^2 + x\rvert$
20.	$\sin x$	$\lvert\sin x\rvert$

Trigonometry

In Exercises 21–24, sketch the graph of the given function. What is the period of the function?

21. $y = \cos 2x$

22. $y = \sin \dfrac{x}{2}$

23. $y = \sin \pi x$

24. $y = \cos \dfrac{\pi x}{2}$

25. Sketch the graph $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

26. Sketch the graph $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 27–30, ABC is a right triangle with the right angle at C. The sides opposite angles A, B, and C are a, b, and c, respectively.

27. a) Find a and b if $c = 2$, $B = \pi/3$.
    b) Find a and c if $b = 2$, $B = \pi/3$.

28. a) Express a in terms of A and c.
    b) Express a in terms of A and b.

29. a) Express a in terms of B and b.
    b) Express c in terms of A and a.

30. a) Express $\sin A$ in terms of a and c.
    b) Express $\sin A$ in terms of b and c.

31. CALCULATOR Two guy wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal, and wire CT makes an angle of 50° with the horizontal, how high is the pole?

32. CALCULATOR Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to

be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

33. Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

34. Express $\cos 3x$ in terms of $\sin x$ and $\cos x$.

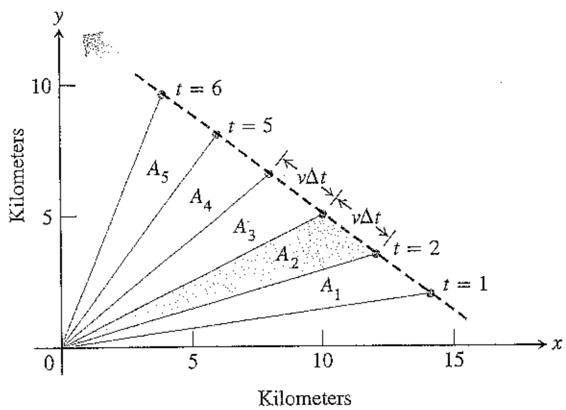
35. a) GRAPHER Graph the function $f(x) = \sin x + \cos(x/2)$.
  b) What appears to be the period of this function?
  c) Confirm your finding in (b) algebraically.

36. a) GRAPHER Graph $f(x) = \sin(1/x)$.
  b) What are the domain and range of f?
  c) Is f periodic? Give reasons for your answer.

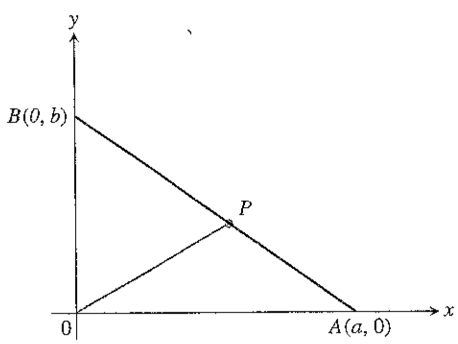
PRELIMINARIES — ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Geometry

1. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 11.5), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



2. a) Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure $(a, b > 0)$.



  b) When is OP perpendicular to AB?

Functions and Graphs

3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ *is* a straight line. Give reasons for your answer.

5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

7. Graph the equation $|x| + |y| = 1 + x$.

8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}$, $A = \pi/3$, $B = \pi/4$.

10. Find $\sin B$ if $a = 4$, $b = 3$, $A = \pi/4$.

11. Find $\cos A$ if $a = 2$, $b = 2$, $c = 3$.

12. Find c if $a = 2$, $b = 3$, $C = \pi/4$.

13. Find $\sin B$ if $a = 2$, $b = 3$, $c = 4$.

14. Find $\sin C$ if $a = 2$, $b = 4$, $c = 5$.

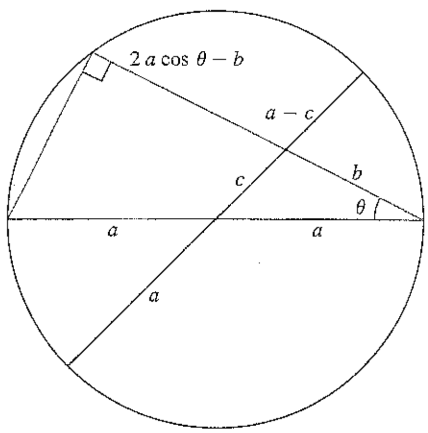
Derivations and Proofs

15. Prove the following identities.

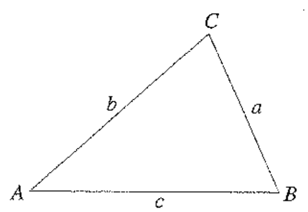
  a) $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

  b) $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine,* Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab\sin C = (1/2)bc\sin A = (1/2)ca\sin B$.



* 18. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter of the triangle.*

19. *Properties of inequalities.* If a and b are real numbers, we say that **a is less than b** and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

If a, b, and c are real numbers, then:

1. $a < b \implies a + c < b + c$
2. $a < b \implies a - c < b - c$
3. $a < b$ and $c > 0 \implies ac < bc$
4. $a < b$ and $c < 0 \implies bc < ac$
   (Special case: $a < b \implies -b < -a$)
5. $a > 0 \implies \dfrac{1}{a} > 0$
6. $0 < a < b \implies \dfrac{1}{b} < \dfrac{1}{a}$
7. $a < b < 0 \implies \dfrac{1}{b} < \dfrac{1}{a}$

20. *Properties of absolute values.* Prove the following properties of absolute values of real numbers.

a) $|-a| = |a|$

b) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

21. Prove that the following inequalities hold for any real numbers a and b.

a) $|a| < |b|$ if and only if $a^2 < b^2$
b) $|a - b| \geq ||a| - |b||$

22. *Generalizing the triangle inequality.* Prove by mathematical induction that the following inequalities hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

a) $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$
b) $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a) *Even-odd decompositions.* Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b) *Uniqueness.* Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

a) a changes while b and c remain fixed?
b) b changes (a and c fixed, $a \neq 0$)?
c) c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x+b)^3 + c$ as

a) a changes while b and c remain fixed?
b) b changes (a and c fixed, $a \neq 0$)?
c) c changes (a and b fixed, $a \neq 0$)?

27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

*Asterisk denotes more challenging problem.