

# Volumes of Solids of Known Cross Sections

## AP Calculus

1. The base of a solid is the region enclosed by a circle centered at the origin with a radius of 5 inches. Find the volume of the solid if all cross sections perpendicular to the  $x$ -axis are squares.

$$x^2 + y^2 = 25 \quad y = \pm \sqrt{25 - x^2}$$

$$A_{\square} = s^2 \quad s = \sqrt{25 - x^2} - (-\sqrt{25 - x^2})$$

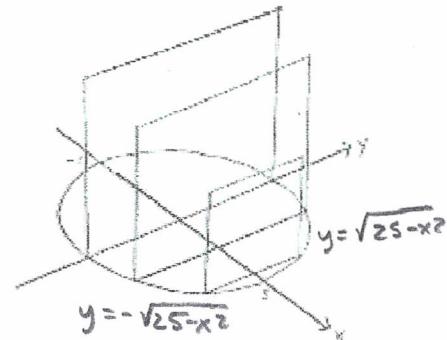
$$s = 2\sqrt{25 - x^2}$$

$$A: (2\sqrt{25 - x^2})^2 = 4(25 - x^2) = 100 - 4x^2$$

$$V = \int_{-5}^5 (100 - 4x^2) dx$$

$$100x - \frac{4x^3}{3} \Big|_{-5}^5$$

$$\left(500 - \frac{500}{3}\right) - \left(-500 + \frac{500}{3}\right) = \frac{1000}{3} + \frac{1000}{3} = \boxed{\frac{2000}{3}} \approx 666.667$$



2. The base of a solid is the circle centered at the origin with a radius of 3 inches. Find the volume of the solid if all cross sections perpendicular to the  $x$ -axis are equilateral triangles.

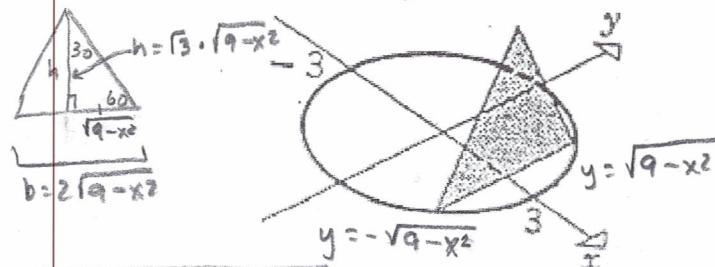
$$A_{\Delta} = \frac{1}{2}bh \quad b = 2\sqrt{9 - x^2} \quad h = \sqrt{3} \cdot \sqrt{9 - x^2}$$

$$A_{\Delta} = \frac{1}{2} \cdot 2\sqrt{9 - x^2} \cdot \sqrt{3} \cdot \sqrt{9 - x^2}$$

$$A_{\Delta} = \sqrt{3}(9 - x^2)$$

$$V = \sqrt{3} \int_{-3}^3 (9 - x^2) dx = \sqrt{3} \left(9x - \frac{x^3}{3}\right) \Big|_{-3}^3$$

$$= \sqrt{3} [(27 - 9) - (-27 + 9)] = \sqrt{3} (18 + 18) = \boxed{36\sqrt{3}} \approx 62.354$$



3. A mathematician has a paperweight made so that its base is the shape of the region between the  $x$ -axis and one arch of the curve  $y = 2 \sin x$ . Each cross-section perpendicular to the  $x$ -axis is a semicircle whose diameter runs from the  $x$ -axis to the curve. Find the volume of the paperweight.

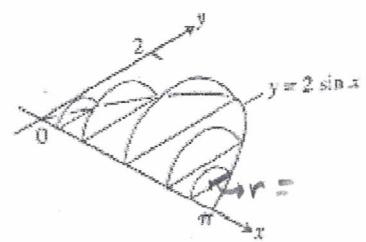
$$A_{\Delta} = \frac{1}{2}\pi r^2 \quad r = \frac{1}{2} \cdot \text{diameter} \quad r = \frac{1}{2} \cdot 2 \sin x \quad r = \sin x$$

$$A = \frac{1}{2}\pi \sin^2 x$$

$$V = \frac{\pi}{2} \int_0^{\pi} \sin^2 x dx = \frac{\pi}{2} \left( \frac{1}{2}x - \frac{1}{4} \sin 2x \right) \Big|_0^{\pi}$$

$$= \frac{\pi}{2} \left[ \left( \frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right) - \left( \frac{1}{2} \cdot 0 - \frac{1}{4} \sin(2 \cdot 0) \right) \right]$$

$$= \frac{\pi}{2} \left[ \left( \frac{\pi}{2} - 0 \right) - (0 - 0) \right] = \boxed{\frac{\pi^2}{4}} \approx 2.467$$



4. The base of a solid is the region in the first quadrant bounded by the graph of  $y = 3x^{\frac{1}{2}} - x^{\frac{3}{2}}$  and the  $x$ -axis.

Cross sections perpendicular to the  $x$ -axis are isosceles right triangles, with one leg in the  $xy$ -plane. What is the volume of the solid?

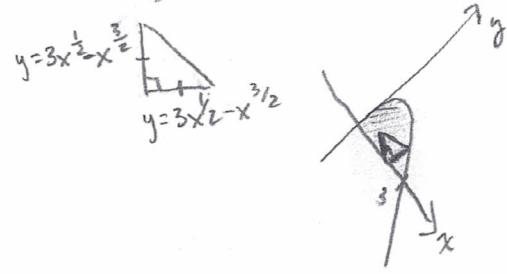
$$A = \frac{1}{2}bh \quad A = \frac{1}{2}(3x^{\frac{1}{2}} - x^{\frac{3}{2}})(3x^{\frac{1}{2}} - x^{\frac{3}{2}})$$

$$A = \frac{1}{2}(9x - 6x^2 + x^3)$$

$$V = \frac{1}{2} \int_0^3 (9x - 6x^2 + x^3) dx = \frac{1}{2} \left( \frac{9x^2}{2} - 2x^3 + \frac{x^4}{4} \right) \Big|_0^3$$

$$= \frac{1}{2} \left[ \left( \frac{81}{2} - 54 + \frac{81}{4} \right) - (0 - 0 + 0) \right]$$

$$= \frac{1}{2} \left( \frac{27}{4} \right) = \boxed{27/8 = 3.375}$$



5. The base of a solid is the elliptical region with boundary curve  $9x^2 + 4y^2 = 36$ . Cross sections perpendicular to the  $x$ -axis are isosceles right triangles with hypotenuse in the base. Find the volume of the solid.

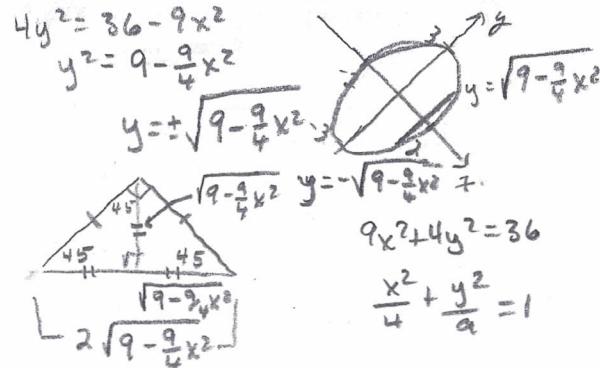
$$A_n = \frac{1}{2}bh \quad b = 2\sqrt{9 - \frac{9}{4}x^2} \quad h = \sqrt{9 - \frac{9}{4}x^2}$$

$$A = \frac{1}{2} \cdot 2\sqrt{9 - \frac{9}{4}x^2} \cdot \sqrt{9 - \frac{9}{4}x^2}$$

$$A = 9 - \frac{9}{4}x^2$$

$$V = \int_{-2}^2 (9 - \frac{9}{4}x^2) dx = 9x - \frac{3}{4}x^3 \Big|_{-2}^2$$

$$V = (18 - 6) - (-18 + 6) = 12 + 12 = \boxed{24}$$

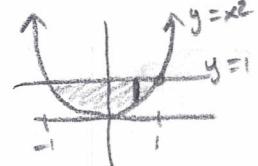


6. The base of a solid is a region bounded by the curves  $y = x^2$  and  $y = 1$ . Cross sections perpendicular to the  $x$ -axis are semicircles. Find the volume of the solid.

$$A = \frac{1}{2}\pi r^2 \quad A = \frac{\pi}{2} \left( \frac{1-x^2}{2} \right)^2 = \frac{\pi}{8} (1-x^2)^2 = \frac{\pi}{8} (1-2x^2+x^4)$$

$$V = \frac{\pi}{8} \int_{-1}^1 (1-2x^2+x^4) dx = \frac{\pi}{8} \left( x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_{-1}^1$$

$$V = \frac{\pi}{8} \left[ (1 - 2/3 + 1/5) - (-1 + 2/3 - 1/5) \right] = \frac{\pi}{8} \left( \frac{8}{15} + \frac{8}{15} \right) = \boxed{\frac{2\pi}{15} \approx 0.419}$$



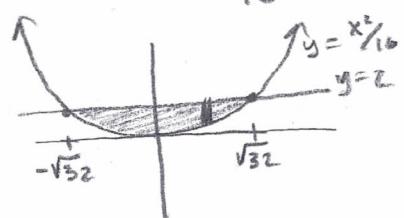
7. The base of a solid is the region bounded by  $x^2 = 16y$  and  $y = 2$ . Cross sections perpendicular to the  $x$ -axis are rectangles whose height is twice that of the side in the  $xy$ -plane. Find the volume of the solid.

$$A = bh \quad A = b \cdot 2b \quad A = 2b^2 \quad b = 2 - \frac{x^2}{16}$$

$$A = 2 \left( 2 - \frac{x^2}{16} \right)^2 = 2 \left( 4 - \frac{x^2}{4} + \frac{x^4}{256} \right) = 8 - \frac{x^2}{2} + \frac{x^4}{128}$$

$$V = \int_{-\sqrt{32}}^{\sqrt{32}} \left( 8 - \frac{x^2}{2} + \frac{x^4}{128} \right) dx = 8x - \frac{x^3}{6} + \frac{x^5}{640} \Big|_{-\sqrt{32}}^{\sqrt{32}}$$

$$V = \left[ \left( 8\sqrt{32} - \frac{\sqrt{32}^3}{6} + \frac{\sqrt{32}^5}{640} \right) - \left( -8\sqrt{32} + \frac{\sqrt{32}^3}{6} - \frac{\sqrt{32}^5}{640} \right) \right] \approx \boxed{48.272}$$



$$\frac{x^2}{16} = 2 \quad x^2 = 32 \quad x = \pm\sqrt{32}$$